Homework 9 Solutions to Additional Problems:

1. Let S be the following ordered basis for $\mathbb{R}^3$. $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   (a) Let $\mathbf{v} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$. Find $[\mathbf{v}]_S$, the coordinate vector of $\mathbf{v}$ with respect to S.

   Setting $\begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, we get the linear system $x + 2z = 4, y + z = -3, x = 2$. This has solution $x = 2, y = -4, z = 1$ so the coordinate vector is $[\mathbf{v}]_S = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$.

   (b) Suppose $\mathbf{w}$ is a vector in $\mathbb{R}^3$ and $[\mathbf{w}]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $\mathbf{w}$.

   $$\mathbf{w} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix}$$

2. Let $S = \{1, t, t^2\}$ and $T = \{t - 1, t^2 + 1, t\}$. These are both ordered bases for P_2. Let $p(t) = 4t^2 - 5t + 3$.

   (a) Find $[p(t)]_S$ and $[p(t)]_T$.

   $p(t) = 3(1) + (-5)(t) + 4(t^2)$ so $[p(t)]_S = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$.

   For $[p(t)]_T$, we need to find x, y, z with $4t^2 - 5t + 3 = x(t-1) + y(t^2+1) + z(t)$. Rewrite this as $4t^2 - 5t + 3 = (y)t^2 + (x + z)t + (-x + y)$ to get the linear system $y = 4, x + z = -5, -x + y = 3$. This has solution $x = 1, y = 4, z = -6$ so $[p(t)]_T = \begin{bmatrix} 1 \\ 4 \\ -6 \end{bmatrix}$.

   (b) Find $P_{S \leftarrow T}$, the transition matrix from T to S.

   To find $P_{S \leftarrow T}$, we take the vectors of T and find their coordinates with respect to S. The first vector in T is $t - 1$. Writing this a linear combination of S

vectors is $t-1=(-1)(1)+(1)(t)+(0)t^2$ so $[t-1]_S = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. The second vector is $t^2+1=(1)(1)+(0)(t)+(1)(t^1)$ so $[t^2+1]_S = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. The third vector is $t=(0)(1)+(1)(t)+(0)(t^2)$ so $[t]_S = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. These three coordinate vectors (in order) are the three columns of $P_{S\leftarrow T}$ so $P_{S\leftarrow T} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(c) Find $Q_{T\leftarrow S}$, the transition matrix from S to T.

Here we take the vectors in S and find their T coordinates. The first vector is 1 so we set $1 = x(t-1)+y(t^2+1)+z(t)$ and which gives the system $y=0, x+z=0, -x+y=1$. This has solution $x=-1, y=0, z=1$ so $[1]_T = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. For the second vector we get $t = x(t-1)+y(t^2+1)+z(t)$. This gives us the system $y=0, x+z=1, -x+y=0$ which has solution $x=0, y=0, z=1$ so $[t]_T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. For the third vector we get $t^2 = x(t-1)+y(t^2+1)+z(t)$. This gives us the system $y=1, x+z=0, -x+y=0$ which has solution $x=1, y=1, z=-1$ so $[t]_T = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. These are the columns of $Q_{T\leftarrow S}$ so $Q_{T\leftarrow S} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

(d) Verify that $[p(t)]_S = P_{S\leftarrow T}[p(t)]_T$ and $[p(t)]_T = Q_{T\leftarrow S}[p(t)]_S$.

$$P_{S\leftarrow T}[p(t)]_T = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} = [p(t)]_S$$

$$Q_{T\leftarrow S}[p(t)]_S = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -6 \end{bmatrix} = [p(t)]_T$$

(e) How are $P_{S\leftarrow T}$ and $Q_{T\leftarrow S}$ related? Hint: multiply them together.

$$P_{S\leftarrow T}Q_{T\leftarrow S} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$P_{S\leftarrow T}$ and $Q_{T\leftarrow S}$ are inverses.

3. For each of the following functions, determine if it is a linear transformation.

(a) $L : \mathbb{R}^3 \to \mathbb{R}^2$ by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} xy \\ z \end{bmatrix}$

This is not a linear transformation. It fails both properties of a linear transformation. Here we will check both properties, but to show it is not a linear transformation you only need to show that one of the two fails.

The first property is $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$. Let $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$.

Then $L(\mathbf{u}+\mathbf{v}) = L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = L\left(\begin{bmatrix} x+x' \\ y+y' \\ z+z' \end{bmatrix}\right) = \begin{bmatrix} (x+x')(y+y') \\ z+z' \end{bmatrix}$.

But $L(\mathbf{u}) + L(\mathbf{v}) = L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + L\left(\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) = \begin{bmatrix} xy \\ z \end{bmatrix} + \begin{bmatrix} x'y' \\ z' \end{bmatrix} = \begin{bmatrix} xy + x'y' \\ z+z' \end{bmatrix}$.

As $(x+x')(y+y') \neq xy + x'y'$ (for example take $x = x' = y = y' = 1$), these are not equal and this property fails.

The second property is $L(r\mathbf{v}) = rL(\mathbf{v})$. Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then $L(r\mathbf{v}) = L\left(r\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = L\left(\begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}\right) = \begin{bmatrix} rxry \\ rz \end{bmatrix}$. The other side is $rL(\mathbf{v}) = rL\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = r\begin{bmatrix} xy \\ z \end{bmatrix} = \begin{bmatrix} rxy \\ z \end{bmatrix}$. Then $rxry \neq rxy$ (for example take $r = 2, x = y = 1$), so these are not equal and this property also fails.

(b) $L : M_{23} \to M_{32}$ by $L(A) = A^T$

This is a linear transformation. Note that taking the transpose swaps the number of rows and columns, so the transpose of a 2×3 matrix is 3×2, so the result of this function applied to things in M_{23} is in M_{32}. To check that it's a linear transformation, check the two properties using the properties of transpose. If A, B are in M_{23} then $L(A+B) = (A+B)^T = A^T + B^T = L(A) + L(B)$

so the first property is satisfied. If A is a 2×3 matrix and r is a scalar, then $L(rA) = (rA)^T = rA^T = rL(A)$ so the second property is also satisfied.

4. Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation $L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x + 2z \\ y - w \\ 3w + z + x \end{bmatrix}$. Find the standard matrix representing L.

   We check what L does to the standard basis for $\mathbb{R}^4$ to get the columns of this matrix.

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},\ L\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

   The standard matrix representing L is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$.