Homework 8 Solutions to Additional Problems:

1. For each of the following set, determine if it is orthogonal, orthonormal, or neither.

   (a) $\left\{ \begin{bmatrix} 2/7 \\ 6/7 \\ -3/7 \end{bmatrix}, \begin{bmatrix} 9/\sqrt{146} \\ 1/\sqrt{146} \\ 8/\sqrt{146} \end{bmatrix} \right\}$

   This set it orthonormal. The dot product is $\begin{bmatrix} 2/7 \\ 6/7 \\ -3/7 \end{bmatrix} \cdot \begin{bmatrix} 9/\sqrt{146} \\ 1/\sqrt{146} \\ 8/\sqrt{146} \end{bmatrix} = \frac{1}{7\sqrt{146}} \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 1 \\ 8 \end{bmatrix} = \frac{1}{7\sqrt{146}}((2)(9) + (6)(1) + (-3)(8)) = 0$ so the two vectors are orthogonal. The length of the first vector is $\frac{1}{7}\sqrt{2^2 + 6^2 + (-3)^2} = \frac{1}{7}\sqrt{49} = 1$ and the length of the second is $\frac{1}{\sqrt{146}}\sqrt{9^2 + 1^2 + 8^2} = 1$. They are both length one so the set is orthonormal.

   (b) $\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \right\}$

   This set is neither. It is not orthogonal because the first and third vector have dot product equal to 1, not 0. It cannot be orthonormal unless it is orthogonal, so it is neither.

   (c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

   This set is orthogonal. The dot product of any two of these is 0 (there are 3 pairs to check) so it is orthogonal. The vectors are length $\sqrt{2}$ though so the set is not orthonormal.

2. Verify that the set $S = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^3$. Use dot products to write the vector $\begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S.

   First check that it's orthogonal.

$$\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}2/\sqrt{6}\\-1/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}=\tfrac{1}{\sqrt{3}\sqrt{6}}((1)(2)+(1)(-1)+(1)(-1))=0$$
$$\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}=\tfrac{1}{\sqrt{3}\sqrt{2}}((1)(0)+(1)(1)+(1)(-1))=0$$
$$\begin{bmatrix}2/\sqrt{6}\\-1/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}\cdot\begin{bmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}=\tfrac{1}{\sqrt{6}\sqrt{2}}((2)(0)+(-1)(1)+(-1)(-1))=0$$
All pairs of distinct vectors are orthogonal, so the set is orthogonal. The lengths are $\frac{1}{\sqrt{3}}\sqrt{1^2+1^2+1^2}=1$, $\frac{1}{\sqrt{6}}\sqrt{2^2+(-1)^2+(-1)^2}=1$ and $\frac{1}{\sqrt{2}}\sqrt{0^2+1^2+(-1)^2}=1$. The vectors are all length 1, so the set is orthonormal.

Any set of nonzero vectors which is orthogonal is linearly independent, so S is a linearly independent set of 3 vectors in $\mathbb{R}^3$ so it is a basis for $\mathbb{R}^3$.

To write $\begin{bmatrix}7\\-2\\1\end{bmatrix}$ as a linear combination, just take its dot product with each of the vectors in S to get the coefficients. The dot products are: $\begin{bmatrix}7\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}=\frac{6}{\sqrt{3}}$, $\begin{bmatrix}7\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}2/\sqrt{6}\\-1/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}=\frac{15}{\sqrt{6}}$, and $\begin{bmatrix}7\\-2\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}=-\frac{3}{\sqrt{2}}$. We thus get that

$$\begin{bmatrix}7\\-2\\1\end{bmatrix}=\frac{6}{\sqrt{3}}\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}+\frac{15}{\sqrt{6}}\begin{bmatrix}2/\sqrt{6}\\-1/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}-\frac{3}{\sqrt{2}}\begin{bmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}.$$

3. Let $\mathbf{v}=\begin{bmatrix}3\\-1\\1\end{bmatrix}$. Let W be the set of all vectors in $\mathbb{R}^3$ which are orthogonal to $\mathbf{v}$. Show that W is a subspace of $\mathbb{R}^3$. Find a basis for W and the dimension of W.

The vectors orthogonal to $\mathbf{v}$ will form a plane though the origin (with normal vector $\mathbf{v}$) so W is a 2-dimensional subspace of $\mathbb{R}^3$.

A vector $\begin{bmatrix}a\\b\\c\end{bmatrix}$ is in W if and only if $3a-b+c=0$. The set W is therefore the set of vectors of the form $\begin{bmatrix} a \\ b \\ -3a+b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ so $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. W is a span of vectors in $\mathbb{R}^3$ so it is a subspace of $\mathbf{R}^3$. The two vectors spanning W are linearly independent so the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for W and W is 2-dimensional.

Note: You can also show W is a subspace by showing it is nonempty and closed under addition and scalar multiplication. However it is probably easier in this case to show that W is a span of vectors and use that spans are always subspaces (since we need to find a basis anyway).

4. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ be a set of vectors in $\mathbb{R}^n$. Suppose that $\mathbf{u}$ is a vector in $\mathbb{R}^n$ which is orthogonal to every vector in S. Is $\mathbf{u}$ orthogonal to every vector in span S? Why or why not?

   Yes. The vectors in span S look like $a_1\mathbf{v_1} + a_2\mathbf{v_2} + ... + a_k\mathbf{v_k}$ with a_i real numbers. Using the properties of dot product, $\mathbf{u} \cdot (a_1\mathbf{v_1} + a_2\mathbf{v_2} + ... + a_k\mathbf{v_k}) = a_1(\mathbf{u} \cdot \mathbf{v_1}) + a_2(\mathbf{u} \cdot \mathbf{v_2}) + ... + a_k(\mathbf{u} \cdot \mathbf{v_k})$. As $\mathbf{u}$ is orthogonal to all the $\mathbf{v_i}$, the terms in this sum are all 0's. Therefore $\mathbf{u} \cdot (a_1\mathbf{v_1} + a_2\mathbf{v_2} + ... + a_k\mathbf{v_k}) = 0$ and $\mathbf{u}$ is orthogonal to any vector in span S.

5. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ with basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   Use Gram-Schmidt process. Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. The new basis will be $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. Take $\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

   Then $\mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{v_1} \cdot \mathbf{u_2}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

Then $\mathbf{v_3} = \mathbf{u_3} - \frac{\mathbf{v_1}\cdot\mathbf{u_3}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{v_2}\cdot\mathbf{u_3}}{\mathbf{v_2}\cdot\mathbf{v_2}}\mathbf{v_2} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \frac{-1}{4}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \frac{1}{4}\left(\begin{bmatrix}4\\4\\4\\0\end{bmatrix} - \begin{bmatrix}4\\4\\0\\0\end{bmatrix} + \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\right) =$ $\frac{1}{4}\begin{bmatrix}1\\-1\\3\\1\end{bmatrix}$.

The orthogonal basis you get is $\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \frac{1}{4}\begin{bmatrix}1\\-1\\3\\1\end{bmatrix}\right\}$. You can also replace $\mathbf{v_3}$ with $4\mathbf{v_3}$ and it will still be an orthogonal basis. So another possible orthogonal basis is $\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\3\\1\end{bmatrix}\right\}$. To get an orthonormal basis, divide each vector by its length to get $\left\{\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{bmatrix}, \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}, \begin{bmatrix}1/\sqrt{12}\\-1/\sqrt{12}\\3/\sqrt{12}\\1/\sqrt{12}\end{bmatrix}\right\}$.