Homework 7 Solutions to Additional Problems:

1. Let W be the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -5 \\ -1 \end{bmatrix} \right\}$. Find a basis for W and $\dim W$.

   We use the method described in the blue box on p.235 of the textbook. If we take a linear combination of these vectors and set it equal to the zero vector, this gives the homogeneous linear system with coefficient matrix $\begin{bmatrix} 1 & 3 & 0 & 4 & 0 \\ 1 & 0 & -3 & 1 & 3 \\ 0 & 2 & 2 & 1 & -5 \\ 0 & 1 & 1 & 1 & -1 \end{bmatrix}$. The row operations $r_2 - r_1 \to r_2, r_2 \leftrightarrow r_4, r_3 - 2r_2 \to r_3, r_4 + 3r_2 \to r_4, -r_3 \to r_3$ give the following matrix in REF $\begin{bmatrix} 1 & 3 & 0 & 4 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Note that you do not need to go all the way to REF or RREF, just far enough to be able to see which columns will contain the leading ones. There are leading ones in columns 1, 2, and 4 so the first, second, and fourth vectors in S will be a basis for the span of S. The basis we get is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. There are 3 vectors in the basis, so the dimension of the span of S is 3.

2. Let V be a finite dimensional vector space and let W be a subspace of V. Prove that if $\dim W = \dim V$, then $W = V$.

   Write n for $\dim W = \dim V$. If $n = 0$, both U and W are equal to the zero vector space so they are equal. Assume now that $n \neq 0$. Let S be a basis for W. Then S is linearly independent and contains exactly n vectors as $\dim W = n$. The vectors in S are also in V so S is a set of n linearly independent vectors in the n-dimensional space V and S is therefore a basis for V (see theorem 4.12a in book). S is a basis for both V and W so $V = \text{span } S$ and $W = \text{span } S$ and hence $W = V$.

3. Let $A = \begin{bmatrix} 0 & 0 & 6 & 0 & 19 & 11 \\ 3 & 12 & 9 & -6 & 26 & 31 \\ 1 & 4 & 3 & -2 & 10 & 9 \\ -1 & -4 & -4 & 2 & -13 & -11 \end{bmatrix}$. The RREF of A is $\begin{bmatrix} 1 & 4 & 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find the rank and the nullity of A.

Both the rank and nullity of A are 3. The rank is 3 as there are 3 columns of the RREF of which contain leading ones (columns 1,3,5). The nullity is 3 as there are 3 columns of the RREF which do not contain leading ones (columns 2,4,6).

(b) Find a basis for the row space of A.

The rank is 3 so any basis for the row space has size 3. The nonzero rows of the RREF are a basis for the row space. This basis is
$\left\{ \begin{bmatrix} 1 & 4 & 0 & -2 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \right\}$.

(c) Find a basis for the column space of A.

The rank is 3 so any basis for the column space also has size 3. The leading ones in the RREF of A are in columns 1, 3, and 5 so the first, third, and fifth columns of A are a basis for the column space. This basis is
$\left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ 9 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 19 \\ 26 \\ 10 \\ -13 \end{bmatrix} \right\}$.

(d) Find a basis for the null space of A.

The nullity of A is 3 so any basis for the null space has size 3. The null space is the solutions to $A\mathbf{x} = \mathbf{0}$. If we label the variables a, b, c, d, e, f then columns 2,4,6 of the RREF of A do not have leading ones so the variables b, d, f can be anything. From the RREF, we get that $e = f, c = -5f, a = -4b + 2d - 4f$ so the null space is all vectors of the form $\begin{bmatrix} -4b + 2d - 4f \\ b \\ -5f \\ d \\ f \\ f \end{bmatrix}$. This can be rewritten as $b\begin{bmatrix}-4\\1\\0\\0\\0\\0\end{bmatrix}+d\begin{bmatrix}2\\0\\0\\1\\0\\0\end{bmatrix}+f\begin{bmatrix}-4\\0\\-5\\0\\1\\1\end{bmatrix}$. The vectors $\left\{\begin{bmatrix}-4\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}2\\0\\0\\1\\0\\0\end{bmatrix},\begin{bmatrix}-4\\0\\-5\\0\\1\\1\end{bmatrix}\right\}$ span the null space and are linearly independent so they are a basis for the null space.

4. Let A be a 5×9 matrix.

(a) Find all possible values for the rank of A.

The possible values for the rank of A are 0,1,2,3,4,5. The rank is equal to the dimension of the row space and there are only 5 rows, so the rank cannot be larger than 5. The rank is also equal to the dimension of the column space, which is a subspace of $\mathbb{R}^5$, so this is another reason it cannot be larger than 5.

(b) Find all possible values for the nullity of A.

The nullity is the number of columns minus the rank. The number of columns is 9, so using the possible values for the rank from part (a), we get that the nullity can be 9,8,7,6,5,4.

(c) If the rows of A are linearly independent, what is the rank of A?

The rank of A would be 5. The row space of A is spanned by the 5 rows of A. If the rows are linearly independent, then they are a basis for the row space of A so the row space would have dimension 5 and the rank is equal to the dimension of the row space.

(d) Are the columns of A linearly independent?

No. If the columns were linearly independent, then the column space would have dimension 9. This is not possible as the rank is equal to the dimension of the column space and the rank is at most 5. Another way to see this is that the columns are 9 vectors in $\mathbb{R}^5$ and any set of more than 5 vectors in $\mathbb{R}^5$ is linearly dependent.

(e) How many solutions does $A\mathbf{x} = \mathbf{0}$ have?

Infinite. A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has 0 or infinite solutions. The nullity of A is not 0 (it is at least 4), so the null space of A is not just the zero vector and hence there are infinite solutions. In general, if A has more columns than rows, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.