Homework 6 Solutions to Additional Problems:

1. Determine if the set $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent. If it is not linearly independent, write one of the vectors in S as a linear combination of the other vectors in S.

   This set is not linearly independent. The third vector is a linear combination of the first two (it is the second one minus two times the first one). We can therefore write the third vector as a linear combination of the other vectors as follows:

$$\begin{bmatrix} 0 \\ -3 \\ 2 \\ -1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

2. Let V be a vector space and let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ be a linearly independent set of vectors in V. Determine if the following sets are linearly independent.

   (a) $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$

   This set is linearly independent (any subset of a linearly independent set is also linearly independent – see Theorem 4.6b in textbook).

   (b) $\{\mathbf{v_1}, \mathbf{v_1} + \mathbf{v_2}, \mathbf{v_1} + \mathbf{v_3}, \mathbf{v_4} - \mathbf{v_2}\}$

   To check if the set is linearly independent, take a linear combination of the vectors and set it equal to the zero vector. So we have $a\mathbf{v_1} + b(\mathbf{v_1} + \mathbf{v_2}) + c(\mathbf{v_1} + \mathbf{v_3}) + d(\mathbf{v_4} - \mathbf{v_2}) = \mathbf{0}$. This equation can be rearranged as $(a + b + c)\mathbf{v_1} + (b - d)\mathbf{v_2} + c\mathbf{v_3} + d\mathbf{v_4} = \mathbf{0}$. As S is linearly independent, the only way for a linear combination of the vectors in S to equal $\mathbf{0}$ is if all the coefficients are 0. It follows that $a + b + c = 0, b - d = 0, c = 0, d = 0$. The only solution to this linear system is $a = b = c = d = 0$. Therefore the vectors in $\{\mathbf{v_1}, \mathbf{v_1} + \mathbf{v_2}, \mathbf{v_1} + \mathbf{v_3}, \mathbf{v_4} - \mathbf{v_2}\}$ are also linearly independent.

   (c) $\{\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}, \mathbf{v_2} - \mathbf{v_3}, \mathbf{v_1} + 2\mathbf{v_2}, \mathbf{v_4}\}$

   Take $a(\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}) + b(\mathbf{v_2} - \mathbf{v_3}) + c(\mathbf{v_1} + 2\mathbf{v_2}) + d\mathbf{v_4} = \mathbf{0}$. This can be rearranged as $(a + c)\mathbf{v_1} + (a + b + 2c)\mathbf{v_2} + (a - b)\mathbf{v_3} + d\mathbf{v_4} = \mathbf{0}$. By the linear independence of S, $a+c=0, a+b+2c=0, a-b=0, d=0$. This linear system has coefficient matrix $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The REF of this matrix is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The corresponding homogeneous linear system has infinite solutions since there is a column without a leading 1, so the vectors are not linearly independent. For example, $a=1, b=1, c=-1, d=0$ would be a linear combination which adds to $\mathbf{0}$ with nonzero coefficients.

Another way to show the set is not linearly independent is by writing one of the vectors as a linear combination of other vectors in the set. In this case, the third vector $\mathbf{v_1}+2\mathbf{v_2}$ is the sum of the first two $\mathbf{v_1}+\mathbf{v_2}+\mathbf{v_3}$ and $\mathbf{v_2}-\mathbf{v_3}$.

3. Let $S=\{\mathbf{v_1},\mathbf{v_2}\}$ be a set of nonzero vectors in a vector space V. Suppose $\mathbf{w}$ is a vector in V and that $\mathbf{w}=5\mathbf{v_1}-\mathbf{v_2}$ and $\mathbf{w}=3\mathbf{v_1}-2\mathbf{v_2}$ (so $\mathbf{w}$ can be written as a linear combination of the vectors in S in more than one way).

   (a) Prove that S is not linearly independent.

   The vector $\mathbf{w}$ is equal to both $5\mathbf{v_1}-\mathbf{v_2}$ and $3\mathbf{v_1}-2\mathbf{v_2}$ and therefore $5\mathbf{v_1}-\mathbf{v_2}=3\mathbf{v_1}-2\mathbf{v_2}$. We can rewrite this equation as $2\mathbf{v_1}+\mathbf{v_2}=\mathbf{0}$. This shows that the vectors are linearly dependent.

   (b) What can you say about the dimension of span S?

   As $\mathbf{v_2}=-2\mathbf{v_1}$, the vectors in the span of S look like $a\mathbf{v_1}+b\mathbf{v_2}=a\mathbf{v_1}+b(-2\mathbf{v_1})=(a-2b)\mathbf{v_1}$. The vectors in span S are therefore multiples of $\mathbf{v_1}$, so span $S=\text{span}\{\mathbf{v_1}\}$. As $\mathbf{v_1}\neq\mathbf{0}$, the set $\{\mathbf{v_1}\}$ is linearly independent and it spans span S, so it is a basis. The dimension of span S is therefore 1 as it has a basis of size 1.

4. Find a basis for and the dimension of each of the following subspaces.

Note: Any vector space V has many different bases. In these solutions, we explain how to find at least one possible basis for each space, but there are many other possible answers that are also correct. The dimension will not depend on the choice of basis however, so there is only one correct answer for the dimension.

(a) The subspace of $\mathbb{R}^5$ which consists of vectors of the form $\begin{bmatrix} 4t+s \\ t-s \\ t \\ 3s \\ s \end{bmatrix}$ where s, t are any real numbers.

Vectors of the form $\begin{bmatrix} 4t+s \\ t-s \\ t \\ 3s \\ s \end{bmatrix}$ can be rewritten as $s \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ so the set of vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ span the space. This set is also linearly independent as the vectors are both nonzero and the second one is not a multiple of the first. This set is therefore a basis for the space and the dimension is 2.

(b) The subspace of M_{23} which consists of matrices of the form $\begin{bmatrix} a-b & c-d & 3a \\ -b & 0 & d-c \end{bmatrix}$ where a, b, c, d are any real numbers.

Matrices of the form $\begin{bmatrix} a-b & c-d & 3a \\ -b & 0 & d-c \end{bmatrix}$ can be rewritten as $a \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + d \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The set

$$\left\{ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

spans the space. It is not linearly independent however. The fourth vector is a multiple of the third, so we can delete the fourth vector without changing the span, and hence the set $\left\{ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$ is also a spanning set for the space. This set is linearly independent so it is a basis and the dimension is 3.

(c) The subspace of P_2 which consists of polynomials at^2+bt+c with $a+b+c=0$.

Given any polynomial at^2+bt+c with $a+b+c=0$, if we solve for $c=-a-b$ and plug this in we get that vectors in this space all have the form $at^2+bt-a-b$.

This can be rewritten as $a(t^2 - 1) + b(t - 1)$ so the set $\{t^2 - 1, t - 1\}$ spans the space. This set is linearly independent so it is a basis and the dimension is 2.

Note: If you solved for a instead of c, you would get the basis $\{-t^2+t, -t^2+1\}$. If you solved for b, you would get $\{t^2 - t, -t + 1\}$. These are also correct answers to this problem.