Homework 5 Solutions to Additional Problems

1. Let V be the set of real numbers and define the operations $\oplus$ and $\odot$ to be the following.

   $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} - 3$ for $\mathbf{u}, \mathbf{v}$ in V
   $r \odot \mathbf{u} = r(\mathbf{u} - 3) + 3$ for $\mathbf{u}$ in V and r a real number.

   Prove that V with the operations $\oplus$ and $\odot$ is a real vector space.

   To do this, we must check all 10 properties (a,b,1-8) from the definition of a vector space.

   Properties a,b (Closed under $\oplus$ and $\odot$): For any $\mathbf{u}, \mathbf{v}$ in V, $\mathbf{u}$ and $\mathbf{v}$ are real numbers so $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} - 3$ is also a real number so $\mathbf{u} \oplus \mathbf{v}$ is in V. This shows V is closed under $\oplus$. For any $\mathbf{u}$ in V and real number r, $r \odot \mathbf{u} = r(\mathbf{u} - 3) + 3$ is also a real number so $r \odot \mathbf{u}$ is in V. This shows V is closed under $\odot$.

   Property 1: $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} - 3$ and $\mathbf{v} \oplus \mathbf{u} = \mathbf{v} + \mathbf{u} - 3$. These are equal.

   Property 2: $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = \mathbf{u} \oplus (\mathbf{v} + \mathbf{w} - 3) = \mathbf{u} + (\mathbf{v} + \mathbf{w} - 3) - 3 = \mathbf{u} + \mathbf{v} + \mathbf{w} - 6$ and $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = (\mathbf{u} + \mathbf{v} - 3) \oplus \mathbf{w} = (\mathbf{u} + \mathbf{v} - 3) + \mathbf{w} - 3 = \mathbf{u} + \mathbf{v} + \mathbf{w} - 6$ so these are equal.

   Property 3: The zero vector is 3 since $\mathbf{u} \oplus 3 = 3 \oplus \mathbf{u} = 3 + \mathbf{u} - 3 = \mathbf{u}$ for all $\mathbf{u}$.

   Property 4: The negative of a vector $\mathbf{u}$ is $(-1)(\mathbf{u} - 6)$ since $\mathbf{u} \oplus (-1)(\mathbf{u} - 6) = \mathbf{u} + (-1)(\mathbf{u} - 6) - 3 = 3$ and 3 is the zero vector.

   Property 5: $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot (\mathbf{u} + \mathbf{v} - 3) = c(\mathbf{u} + \mathbf{v} - 3 - 3) + 3 = c\mathbf{u} + c\mathbf{v} - 6c + 3$ and $c \odot \mathbf{u} \oplus c \odot \mathbf{v} = (c(\mathbf{u} - 3) + 3) \oplus (c(\mathbf{v} - 3) + 3) = (c(\mathbf{u} - 3) + 3) + (c(\mathbf{v} - 3) + 3) - 3 = c\mathbf{u} + c\mathbf{v} - 6c + 3$ so these are equal.

   Property 6: $(c + d) \odot \mathbf{u} = (c + d)(\mathbf{u} - 3) + 3$ and $c \odot \mathbf{u} \oplus d \odot \mathbf{u} = (c(\mathbf{u} - 3) + 3) \oplus (d(\mathbf{u} - 3) + 3) = (c(\mathbf{u} - 3) + 3) + (d(\mathbf{u} - 3) + 3) - 3 = (c + d)(\mathbf{u} - 3) + 3$ so these are equal.

   Property 7: $c \odot (d \odot \mathbf{u}) = c \odot (d(\mathbf{u} - 3) + 3) = c(d(\mathbf{u} - 3) + 3 - 3) + 3 = cd(\mathbf{u} - 3) + 3 = (cd) \oplus \mathbf{u}$

   Property 8: $1 \odot \mathbf{u} = 1(\mathbf{u} - 3) + 3 = \mathbf{u}$.

2. Determine which of the following are subspaces. You may assume the operations are the usual addition and scalar multiplication in $\mathbb{R}^n$ and P.

(a) Let V be the set of 2-vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $|y| = |x|$. Is V a subspace of $\mathbb{R}^2$?

This is not a subspace. It is not closed under addition. For example, the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are in V but their sum is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ which is not in V.

Note: V is closed under scalar multiplication. We do not need to check this however since the fact that it's not closed under addition already shows it is not a subspace of $\mathbb{R}^2$.

(b) Let V be the set of polynomials $p(t)$ such that $\int_0^1 p(t)\ dt = 0$. Is V a subspace of P?

This is a subspace of P. To prove this, we check that V is not the empty set and that V is closed under addition and scalar multiplication.

V is not the empty set because it contains the zero vector of P (the function $p(t) = 0$ which is 0 everywhere).

To check if it is closed under addition, take $p(t)$ and $q(t)$ to be polynomials in V and see if $p(t) + q(t)$ is in V. As $p(t)$ and $q(t)$ are in V, we have that $\int_0^1 p(t)\ dt = 0$ and $\int_0^1 q(t)\ dt = 0$. Using properties of integrals, $p(t) + q(t)$ is also in V because $\int_0^1 p(t) + q(t)\ dt = \int_0^1 p(t)\ dt + \int_0^1 p(t)\ dt = 0 + 0 = 0$. V is therefore closed under addition.

To check if V is closed under scalar multiplication, take $p(t)$ to be a polynomial in V and r to be a real number. As $p(t)$ is in V, $\int_0^1 p(t)\ dt = 0$. The the scalar multiple $rp(t)$ is also in V because $\int_0^1 rp(t)\ dt = r\int_0^1 p(t)\ dt = r0 = 0$. This shows that V is closed under scalar multiplication.

(c) Let V be the set of polynomials $p(t)$ such that $p(0) = 5$. Is V a subspace of P?

This is not a subspace of P. It is not closed under addition or scalar multiplication. If $p(t)$ and $q(t)$ are in V, then $p(0) = 5$ and $q(0) = 5$. Their sum $p(t) + q(t)$ will be 10 when $t = 0$, so it will not be in V. For example, $t + 5$ and $t^2 + 3t + 5$ are both in V, but their sum $t^2 + 4t + 10$ is not.

You can also show this is not a subspace by showing that is not closed under scalar multiplication, or by showing that it does not contain the zero vector.

(d) Let A be a fixed 3×3 matrix. Let V be the set of 3-vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is a consistent linear system. Is V a subspace of $\mathbb{R}^3$?

This is a subspace. We first note that V is nonempty as $A\mathbf{x} = \mathbf{0}$ is consistent so $\mathbf{0}$ is in V.

If $\mathbf{b}$ and $\mathbf{c}$ are in V, then the systems $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{c}$ are both consistent. This means that they both have at least one solution (possibly different). Let $\mathbf{v_1}$ be a solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v_2}$ be a solution to $A\mathbf{x} = \mathbf{c}$. Then $A\mathbf{v_1} = \mathbf{b}$ and $A\mathbf{v_2} = \mathbf{c}$. The linear system $A\mathbf{x} = \mathbf{b}+\mathbf{c}$ has solution $\mathbf{v_1}+\mathbf{v_2}$ as $A(\mathbf{v_1}+\mathbf{v_2}) = A\mathbf{v_1}+A\mathbf{v_2} = \mathbf{b}+\mathbf{c}$. Then $A\mathbf{x} = \mathbf{b}+\mathbf{c}$ has at least one solution, so it is consistent and therefore $\mathbf{b}+\mathbf{c}$ is in V. This shows that V is closed under addition.

If $\mathbf{b}$ is in V and r is a real number, then the systems $A\mathbf{x} = \mathbf{b}$ is consistent. Let $\mathbf{v_1}$ be a solution to $A\mathbf{x} = \mathbf{b}$, so $A\mathbf{v_1} = \mathbf{b}$. The linear system $A\mathbf{x} = r\mathbf{b}$ has solution $r\mathbf{v_1}$ as $A(r\mathbf{v_1}) = r(A\mathbf{v_1}) = r\mathbf{b}$. Then $A\mathbf{x} = r\mathbf{b}$ has at least one solution, so it is consistent and therefore $r\mathbf{b}$ is in V. This shows that V is closed under scalar multiplication.

3. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\}$. Does S span $\mathbb{R}^3$? Either prove that S spans $\mathbb{R}^3$, or find a vector in $\mathbb{R}^3$ which is not in the span of S.

The span of S is all linear combinations of vectors in S, so it is all vectors of the form $x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + z \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + w \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$, where x, y, z, w are real numbers. Given any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$, to see if it's in the span of S we want to see if $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + z \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + w \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ for some x, y, z, w. This gives us the linear system with augmented matrix $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & a \\ 1 & 0 & 1 & 1 & b \\ 1 & 2 & 3 & 3 & c \end{array}\right]$. The row operations $r_2 - r_1 \to r_2, r_3 - r_1 \to r_3, r_2 + r_3 \to r_2, r_2 \leftrightarrow r_3, (-1/2)r_3 \to r_3$ take this to the matrix

$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & a \\ 0 & 1 & 1 & 1 & c-a \\ 0 & 0 & 0 & 1 & (-1/2)(b+c-2a) \end{array}\right]$, which is in REF. No matter what we choose for a, b, c, this system will always have infinitely many solutions. Therefore all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ are in the span of S, so S spans $\mathbb{R}^3$.

4. Let W be the set of 3×3 skew symmetric matrices. Find a set S of 3×3 matrices such that $W = \text{span } S$. Is W a subspace of M_{33}?

The 3×3 skew symmetric matrices are all matrices of the form $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$, where a, b, c can be any real numbers. We can rewrite these matrices as $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. From this, we see that all matrices in W are linear combinations of the set of matrices

$$S = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}.$$

Therefore S is a spanning set for W. Note that spanning sets are not unique and there are lots of other possible answers for this problem, but this set is perhaps the easiest one to find.

W is a subspace of M_{33} because $W = \text{span } S$, where S is a set of elements from M_{33}.