Homework 4 Solutions to Additional Problems:

1. Let $A = \begin{bmatrix} 2 & 2 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 6 & 0 & 0 & 7 \\ 0 & 0 & -3 & 0 \end{bmatrix}$. Compute the determinant of A using three different methods.

   (a) The definition of the determinant

   The nonzero entries in the definition of the determinant are products of 4 nonzero entries from A with one entry in each row and column. The only nonzero entry in row 4 is the -3 in column 3, so that will have to be picked. Then we have cannot take any more entries from column 3 so in row 2 we must pick the 1 in column 2. There are two possible ways to pick the remaining entries.
   $$A = \begin{bmatrix} 2 & 2 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 6 & 0 & 0 & 7 \\ 0 & 0 & -3 & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 2 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 6 & 0 & 0 & 7 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$
   There are therefore 2 nonzero terms in the sum. The first one is $a_{11}a_{22}a_{34}a_{43} = (2)(1)(7)(-3) = -42$. The corresponding permutation can be gotten by going through the rows in order and taking the column that has the circled entry. So for this one, the permutation is 1243. This has one inversion (43) so it is odd and the term gets a $(-)$. The second one is $a_{14}a_{22}a_{31}a_{43} = (1)(1)(6)(-3) = -18$. This has permutation 4213 which has four inversions $(21, 42, 41, 43)$ so it is even and gets a $(+)$. Putting these together, we get that $\det(A) = (-)(-42) + (+)(-18) = 42 - 18 = 24$.

   (b) Reduction to triangular form

   The row operations $r_3 - 3r_1 \to r_3, r_3 + 6r_2 \to r_3, r_3 \leftrightarrow r_4$ will take A to the matrix $B = \begin{bmatrix} 2 & 2 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ which is an upper triangular matrix. $\det(B)$ is the product of the diagonal entries so $\det(B) = (2)(1)(-3)(4) = -24$. The first two row operations were type 3, so they did not change the determinant. The third row operation was a type 1 row operation so it changed the sign of the determinant. Therefore $-\det(A) = \det(B)$ so $\det(A) = 24$.

(c) Cofactor expansion

There are a lot of possible ways to do this depending on the choices of which row and columns to expand along. Here we will do the one with the least number of terms. Expanding along row 4 we get $\det(A) = (-)(-3)\det\left(\begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 6 & 0 & 7 \end{bmatrix}\right)$.

We next expand along row 2, so $\det(A) = (-)(-3)(+)(1)\det\left(\begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix}\right) = 3(14-6) = 3(8) = 24$.

2. Suppose A and B are 3×3 matrices such that $B^{-1} = \frac{1}{5}A^2$. If $\det(A) = 2$, what is $\det(B)$?

As $B^{-1} = \frac{1}{5}A^2$, we take determinants of both sides to get $\det(B^{-1}) = \det\left(\frac{1}{5}A^2\right)$. By properties of determinants, $\det(B^{-1}) = \frac{1}{\det(B)}$ and $\det\left(\frac{1}{5}A^2\right) = \left(\frac{1}{5}\right)^3\det(A)^2 = \frac{4}{125}$. The previous equation becomes $\frac{1}{\det(B)} = \frac{4}{125}$ so $\det(B) = \frac{125}{4}$.

3. Let A be a square matrix. Determine if the following are true or false. Give a proof or counterexample.

(a) If $AA^T = I$, then $\det(A) = \pm 1$.

True. If $AA^T = I$, then $\det(AA^T) = \det(I)$. By the properties of determinant, $\det(AA^T) = \det(A)\det(A^T) = \det(A)^2$. Also $\det(I) = 1$. We therefore get that $\det(A)^2 = 1$ so $\det(A) = \pm 1$.

(b) If $A + A^T = I$, then $\det(A) = \frac{1}{2}$.

False. If $A + A^T = I$, then $\det(A + A^T) = \det(I)$. However there is no nice property for the determinant of a sum of matrices, so it appears that there is no way to find $\det(A)$ from the given information. We therefore suspect it might be false and look for a counterexample. If A is a 2×2 matrix, then the equation $A + A^T = I$ forces A to look like $A = \begin{bmatrix} 1/2 & b \\ -b & 1/2 \end{bmatrix}$ for some b. This matrix has determinant $\frac{1}{4} + b^2$ so we see that the determinant does not have to be $\frac{1}{2}$ and the statement is false. In particular, the matrix $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ is a counterexample as it satisfies the equation $A + A^T = I$ but does not have determinant $\frac{1}{2}$.

4. Let A be a 5×5 matrix with determinant 5.

   (a) What is the RREF of A?

   As $\det(A) \neq 0$, A is invertible so its RREF is the 5×5 identity matrix I_5.

   (b) Let $\mathbf{b} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$. How many solutions are there to the linear system $A\mathbf{x} = \mathbf{b}$?

   A is invertible so for any 5-vector $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, $\mathbf{x} = A^{-1}\mathbf{b}$.