Homework 3 Solutions to Additional Problems:

1. Let A and B be $n \times n$ upper triangular matrices. Determine if the following matrices are upper triangular, lower triangular, both, or neither.

   (a) $(A+B)^T$

   This is lower triangular. The sum $A+B$ is upper triangular, then the transpose of an upper triangular matrix is lower triangular.

   (b) AB

   This is upper triangular. The i,j-th entry of AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$. Suppose we are looking at an entry below the diagonal, so $i > j$. Then if $k < i$, the term $a_{ik} = 0$ and if $k \geq i > j$ the term $b_{kj} = 0$ (using that A and B are upper triangular). Therefore all entries in the sum are 0 so the i,j-th entry of AB is 0 for $i > j$ and AB is upper triangular.

   (c) AB^T

   This is neither. For example, $A = B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has $AB^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ which is neither upper triangular nor lower triangular.

2. Let $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

   (a) Compute D^2 and D^3.

   $$D^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \; D^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

   (b) Compute D^{100}. You do not need to simplify the entries of D^{100}.

   $$D^{100} = \begin{bmatrix} 1^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix}.$$

   (c) If E is an $n \times n$ diagonal matrix with diagonal entries $e_1, e_2, ..., e_n$, what can you say about the matrix E^k for k a positive integer?

E^k is also an $n \times n$ diagonal matrix. The diagonal entries of E^k are ${e_1}^k, {e_2}^k, ..., {e_n}^k$.

3. Let A be an $n \times n$ matrix

   (a) Prove that $A + A^T$ is symmetric and $A - A^T$ is skew symmetric.

   To show that $A + A^T$ is symmetric, show that its transpose is equal to itself. Using the properties of transpose, $(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$. To show that $A - A^T$ is skew symmetric, show that its transpose it equal to its negative. $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$.

   (b) Is the matrix $(A + A^T)(A - A^T)$ symmetric, skew symmetric, both, or neither? Either give a proof that it is always symmetric, skew symmetric, or both or find a specific example of a matrix A for which $(A+A^T)(A-A^T)$ is neither.

   This is neither. If we try taking the transpose of this matrix, we get $((A + A^T)(A-A^T))^T = (A-A^T)^T(A+A^T)^T = -(A-A^T)(A+A^T)$. It almost looks like it will be skew symmetric, but not quite since $(A+A^T)(A-A^T)$ may not be the same as $(A-A^T)(A+A^T)$. Since it is not obviously symmetric or skew symmetric, try an example. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. Then $A + A^T = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$ and $A - A^T = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, so $(A+A^T)(A-A^T) = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -12 & 4 \end{bmatrix}$. We see from this example that $(A + A^T)(A - A^T)$ is not always symmetric, skew symmetric, or both.

4. Suppose A and B are invertible 3×3 matrices and that $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & 3 \\ 1 & 7 & 5 \end{bmatrix}$. Answer the following without computing A^{-1} or B.

   (a) Find $(A^{-1}B^T)^{-1}$.

   Using the properties for inverses, $(A^{-1}B^T)^{-1} = (B^T)^{-1}(A^{-1})^{-1} = (B^{-1})^T A$. We can easily compute this from the given information by taking one transpose and one product. $(A^{-1}B^T)^{-1} = (B^{-1})^T A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 7 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ -1 & 1 & 0 \end{bmatrix} =$

$$\begin{bmatrix} -4 & -2 & 0 \\ -7 & 7 & 0 \\ 3 & 20 & 7 \end{bmatrix}.$$

(b) Let $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. Find all solutions to the linear system $A^{-1}B^T\mathbf{x} = \mathbf{c}$.

If we multiply both sides of the equation $A^{-1}B^T\mathbf{x} = \mathbf{c}$ on the left by $(A^{-1}B^T)^{-1}$ we get $I\mathbf{x} = (A^{-1}B^T)^{-1}\mathbf{c}$ which simplifies to $\mathbf{x} = (A^{-1}B^T)^{-1}\mathbf{c}$. There is therefore exactly one solution, it is $(A^{-1}B^T)^{-1}\mathbf{c}$. Using the result from part (b), this is $\begin{bmatrix} -4 & -2 & 0 \\ -7 & 7 & 0 \\ 3 & 20 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 9 \end{bmatrix}$.

5. Let $A = \begin{bmatrix} -2 & 0 & -5 & 2 \\ 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 5 & 0 \end{bmatrix}$. Find A^{-1} using the methods of Section 2.3. Check your answer by computing AA^{-1} or $A^{-1}A$.

Start with $[A : I]$ and do row operations to get to $[I : A^{-1}]$.

$$\left[\begin{array}{cccc:cccc} -2 & 0 & -5 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$r_1 \leftrightarrow r_2$

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 3 & -1 & 0 & 1 & 0 & 0 \\ -2 & 0 & -5 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 5 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$r_2 + 2r_1 \rightarrow r_2$
$r_4 - r_1 \rightarrow r_4$

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$r_2 \leftrightarrow r_3$

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$r_4 - r_2 \rightarrow r_4$

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \end{array}\right]$$

$r_1 + r_4 \rightarrow r_1$

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 3 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \end{array}\right]$$

$r_1 - 3r_3 \rightarrow r_1$
$r_2 - 2r_3 \rightarrow r_2$

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & -3 & -6 & -1 & 1 \\ 0 & 1 & 0 & 0 & -2 & -4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -3 & -6 & -1 & 1 \\ -2 & -4 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}.$$

Check: $AA^{-1} = \begin{bmatrix} -2 & 0 & -5 & 2 \\ 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & -6 & -1 & 1 \\ -2 & -4 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I.$