Solutions to Homework 2 Additional Problems

1. Solve each linear system using Gaussian elimination or Gauss-Jordan reduction.

   (a) $2x + y = 8$
   $3x + 4y = 7$
   $x + y = 3$

   The augmented matrix of this linear system is $\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 3 & 4 & 7 \\ 1 & 1 & 3 \end{array}\right]$. The row operations $r_1 \leftrightarrow r_3, r_2 - 3r_1 \to r_2, r_3 - 2r_1 \to r_3, r_3 + r_2 \to r_2$ take the augmented matrix to the matrix $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$, which is in REF.

   Using Gaussian elimination: The matrix in REF is the system $x + y = 3, y = -2$. Substituting $y = -2$ into the first equation gives $x = 5$ so the only solution is $x = 5, y = -2$. Written as a vector this is $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

   Using Gauss-Jordan reduction: The row operation $r_1 - r_2 \to r_1$ will take the matrix in REF to the matrix $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$, which is in RREF. This system is $x = 5, y = -2$ so the only solution is $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

   (b) $x + y + z = 3$
   $2x + 2y - z = 3$

   The augmented matrix of this linear system is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & -1 & 3 \end{array}\right]$. The row operations $r_2 - 2r_1 \to r_2, (-1/3)r_2 \to r_2$ take the augmented matrix to the matrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$, which is in REF.

   Using Gaussian elimination: The matrix in REF is the system $x + y + z = 3, z = 1$. Substituting $z = 1$ into the first equation gives $x + y = 2$. The column corresponding to y does not have a leading 1, so y can be anything and $x = 2 - y$. The solutions are all vectors of the form $\begin{bmatrix} 2 - y \\ y \\ 1 \end{bmatrix}$.

Using Gauss-Jordan reduction: The row operation $r_1 - r_2 \to r_1$ will take the matrix in REF to the matrix $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$, which is in RREF. This system is $x + y = 2, z = 1$. The column corresponding to y does not have a leading 1, so y can be anything and $x = 2 - y$. The solutions are all vectors of the form $\begin{bmatrix} 2-y \\ y \\ 1 \end{bmatrix}$.

(c) $2x + y + z = 1$
$4x - 3y + z = 7$
$3x - y + z = 2$

The augmented matrix for this system is $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \\ 3 & -1 & 1 & 2 \end{array}\right]$. The row operations $r_3 - r_1 \to r_3, r_1 \leftrightarrow r_3, r_2 - 2r_1 \to r_2, r_3 - 4r_1 \to r_3, r_3 - r_2 \to r_3$ get the matrix $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 5 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{array}\right]$. This is not yet in REF, but we can see from row 3 that there are no solutions.

2. Let A be an $m \times n$ matrix and $\mathbf{b}$ be an m-vector such that $A\mathbf{x} = \mathbf{b}$ is a consistent linear system. Let $\mathbf{x}_p$ be a solution to $A\mathbf{x} = \mathbf{b}$ and let $\mathbf{0}$ be the zero m-vector.

(a) Prove that if $\mathbf{x}_h$ is a solution to $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_p + \mathbf{x}_h$ is a solution to $A\mathbf{x} = \mathbf{b}$.

If $\mathbf{x}_h$ is a solution to $A\mathbf{x} = \mathbf{0}$, then $A\mathbf{x}_h = \mathbf{0}$. Also, since $\mathbf{x}_p$ is a solution to $A\mathbf{x} = \mathbf{b}$ this means that $A\mathbf{x}_p = \mathbf{b}$. We want to use these facts to prove that $\mathbf{x}_p + \mathbf{x}_h$ is a solution to $A\mathbf{x} = \mathbf{b}$. To do this, we just need to plug in $\mathbf{x}_p + \mathbf{x}_h$ for $\mathbf{x}$ in the equation $A\mathbf{x} = \mathbf{b}$ and show that the equation still holds. Using the properties of matrix operations, $A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$, so $\mathbf{x}_p + \mathbf{x}_h$ is a solution to $A\mathbf{x} = \mathbf{b}$.

(b) Prove that if $\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_p$ is a solution to $A\mathbf{x} = \mathbf{0}$.

If $\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{x}_1 = \mathbf{b}$. As in the previous part, we know that $A\mathbf{x}_p = \mathbf{b}$. To prove that $\mathbf{x}_1 - \mathbf{x}_p$ is a solution to $A\mathbf{x} = \mathbf{0}$, we plug in $\mathbf{x}_1 - \mathbf{x}_p$ for $\mathbf{x}$ in the equation $A\mathbf{x} = \mathbf{0}$ and show that the equation still holds. Using the properties of matrix operations, $A(\mathbf{x}_1 - \mathbf{x}_p) = A\mathbf{x}_1 - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}$, so $\mathbf{x}_1 - \mathbf{x}_p$ is a solution to $A\mathbf{x} = \mathbf{0}$.

(c) Use the previous two parts to explain why the following statement is true: The solutions to $A\mathbf{x} = \mathbf{b}$ are exactly the vectors of the form $\mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_h$ is a solution to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

We know from part (a) that any vector of the form $\mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_h$ is a solution to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is a solution to $A\mathbf{x} = \mathbf{b}$. It remains to show that there are no other solutions that do not have this form. In other words, we need to show given any solution, we can always write it in the form $\mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_h$ is a solution to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. If $\mathbf{x}_1$ is any solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 = \mathbf{x}_p + (\mathbf{x}_1 - \mathbf{x}_p)$. By part (b), the vector $\mathbf{x}_1 - \mathbf{x}_p$ is a solution to $A\mathbf{x} = \mathbf{0}$ so we take $\mathbf{x}_h = \mathbf{x}_1 - \mathbf{x}_p$ and we have shown that $\mathbf{x}_1$ has the form $\mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_h$ is a solution to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

Note: The p and h used as subscripts in this problem stand for particular and homogeneous.

3. Let $A = \begin{bmatrix} 3 & 1 & -1 & -2 & 2 \\ 1 & 1 & 1 & 1 & 8 \\ 7 & -1 & -9 & -4 & 4 \\ 5 & 3 & 1 & -5 & -2 \end{bmatrix}$. The RREF of A is $\begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ -3 \\ 8 \end{bmatrix}$.

(a) Find all solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

We will label the RREF of A as B. The augmented matrix of the homogeneous linear system looks like $[A : \mathbf{0}]$. Note that the last column is all zeros and doing row operations will not change a column of zeros. It follows that the RREF of $[A : \mathbf{0}]$ is $[B : \mathbf{0}]$. Therefore the solutions to $A\mathbf{x} = \mathbf{0}$ are the same as the solutions to $B\mathbf{x} = \mathbf{0}$. Then $[B : \mathbf{0}] = \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. If we label the variables a, b, c, d, e we see that the columns corresponding to c, e do not contain leading ones. Therefore c, e are free variables and the other variables can be solved in terms of these variables. The fourth equation is $0 = 0$ which is true but gives us no information. The third equation is $d + 4e = 0$ so $d = -4e$. The second equation is $b + 2c + e = 0$ so $b = -2c - e$. The first equation is $a - c + 3e = 0$ so $a = c - 3e$. Putting this all together, we get that the solutions to the homogeneous linear system are all vectors of the form $\begin{bmatrix} c-3e \\ -2c-e \\ c \\ -4e \\ e \end{bmatrix}$, where c, e can be any real numbers.

Note: To check this answer, you can multiply by A and make sure everything cancels to give you the zero vector.

(b) Prove that $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ is a solution to the linear system $A\mathbf{x} = \mathbf{b}$.

Plug in $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ for $\mathbf{x}$ in $A\mathbf{x} = \mathbf{b}$ and see if the equation holds. $A \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} =$

$$\begin{bmatrix} 3 & 1 & -1 & -2 & 2 \\ 1 & 1 & 1 & 1 & 8 \\ 7 & -1 & 9 & -4 & 4 \\ 5 & 3 & 1 & -5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3(1) + 1(2) + (-1)(0) + (-2)(1) + 2(-1) \\ 1(1) + 1(2) + 1(0) + 1(1) + 8(-1) \\ 7(1) + (-1)(2) + (-9)(0) + (-4)(1) + 4(-1) \\ 5(1) + 3(2) + 1(0) + (-5)(1) + (-2)(-1) \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ -4 \\ -3 \\ 8 \end{bmatrix} = \mathbf{b}.$$

(c) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

Hint: Use the result from problem 2c.

This has augmented matrix $[A : \mathbf{b}]$. The RREF is NOT going to be $[B : \mathbf{b}]$, because the row operations used to get from A to B will change $\mathbf{b}$. If we wanted to solve this by finding RREF of $[A : \mathbf{b}]$, we would have to figure out the row operations used to get from A to B and do them to $\mathbf{b}$. However we can avoid doing any row operations by using Problem 2c and the results from parts (a) and (b) of this problem. By Problem 2c, we know that if we have a particular solution $\mathbf{x}_p$ to the linear system $A\mathbf{x} = \mathbf{b}$, then the solutions to $A\mathbf{x} = \mathbf{b}$ are all vectors that look like $\mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_h$ is a solution to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. By part (b), we can take $\mathbf{x}_p = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

By part (a), the solutions to the homogeneous system are all vectors of the form $\begin{bmatrix} c - 3e \\ -2c - e \\ c \\ -4e \\ e \end{bmatrix}$. The solutions to $A\mathbf{x} = \mathbf{b}$ are therefore all vectors of the form $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} c - 3e \\ -2c - e \\ c \\ -4e \\ e \end{bmatrix}$, or written as single vector, all vectors of the form $\begin{bmatrix} 1 + c - 3e \\ 2 - 2c - e \\ c \\ 1 - 4e \\ -1 + e \end{bmatrix}$.