Homework 1: Solutions to Additional Problems

(1)
1) $x+y+z=-1$
2) $x-y+3z=c^2$
3) $x+2z=4$

For what value(s) of c is the system consistent?

If we add eq 1 and eq 2 we get $2x+4z=-1+c^2$,

and dividing by 2 we get $x+2z=\frac{-1+c^2}{2}$.

The only way that this equation and equation 3 $(x+2z=4)$ both hold is if $\frac{-1+c^2}{2}=4$

$$-1+c^2=8$$
$$c^2=9$$
$$c=\pm 3$$

Therefore if $c\neq \pm 3$, the system is not consistent.

If $c=\pm 3$, the system becomes
$$x+y+z=-1$$
$$x-y+3z=9$$
$$x+2z=4.$$

The solutions to this system are:

$$\left.\begin{array}{l} z=\text{anything} \\ x=4-2z \\ y=-5+z \end{array}\right\}$$ infinite solutions, so consistent when $c=\pm 3$

Answer: $c=\pm 3$

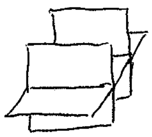
(2) Describe the possible numbers of solutions to a linear system of 3 equations + 3 unknowns.

Each equation gives you a plane in 3-dimensional space.
The solutions are the points on all 3 planes, so think about the ways that 3 planes can intersect.

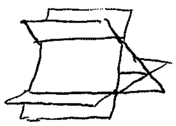
Suppose all 3 planes are different. There are 5 possibilities for the ways they intersect.

1) 

All 3 parrallel - no points of intersection so 0 solutions

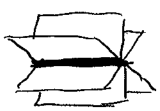
2) 

2 planes parallel - the 2 parrallel planes each intersect the 3rd plane in lines, but there are no points where all 3 planes intersect so 0 solutions

3) 

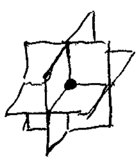
Again, no points where all 3 planes intersect

0 solutions

4) 

The 3 planes intersect in a line. There are infinitely many points on a line.

Infinite solutions

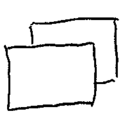
5) 

Exactly 1 point on all 3 planes

1 solution

If 2 of the equations give you the same plane $(P_1 = P_2)$ and the 3rd equation is a different plane P_3, then the solutions are intersections of 2 planes ($P_1 = P_2$ and P_3)

2 possibilities

1) 

parallel - 0 points of intersection

0 solutions

2) 

Intersection is a line

Infinite solutions

Finally, if all 3 equations are the same plane, intersection is a plane so contains infinite points and there are infinite solutions.

Answer: 0, 1, or infinite solutions

3) $2x + y - z + w = 0$

$-x - 2y + z - 2w = 0$

a) Show $x_1 = 1, y_1 = -2, z_1 = 3, w_1 = 3$ and $x_2 = 0, y_2 = 1, z_2 = 0, w_2 = -1$ are solutions

$2(1) + (-2) - 3 + 3 = 0$ ✓

$-(1) - 2(-2) + 3 - 2(3) = 0$ ✓

$2(0) + 1 - 0 + (-1) = 0$ ✓

$-(0) - 2(1) + 0 - 2(-1) = 0$ ✓

b) Is $x_3 = x_1 + x_2, y_3 = y_1 + y_2, z_3 = z_1 + z_2, w_3 = w_1 + w_2$ a solution?

$x_3 = 1$

$y_3 = -1$

$z_3 = 3$

$w_3 = +2$

$2(1) + (-1) - 3 + (+2) = 0$

$-(1) - 2(-1) + 3 - 2(2) = 0$

yes, it is a solution

c) Is $x_4 = 5x_1, y_4 = 5y_1, z_4 = 5z_1, w_4 = 5w_1$ a solution?

$x_4 = 5 \cdot 1$

$y_4 = 5 \cdot (-2)$

$z_4 = 5(3)$

$w_4 = 5(3)$

$2(5 \cdot 1) + 5(-2) - 5(3) + 5(3) = 5(2 \cdot 1 - 2 - 3 + 3) = 5 \cdot 0 = 0$

$-(5 \cdot 1) - 2(5(-2)) + 5(3) - 2(5 \cdot 3) = 5(-1 - 2(-2) + 3 - 2(3)) = 5 \cdot 0 = 0$

yes, it is a solution

4) a) Find coefficient + augmented matrices of the system

$x + 3y - z + w = 4$

$2x + 2w = 1$

$z - 6w = 0$

coefficient matrix

$$\begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 4 \\ 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -6 & 0 \end{array}\right]$$

b) Find the linear system w/ augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 3 & 3 & 4 & -2 \end{array}\right]$$

linear system

$$\begin{cases} x + 2y = 1 \\ 3x + 3y + 4z = -2 \end{cases}$$

⑤ Find RREF

$$\begin{bmatrix} 0 & 1 & 2 & 5 \\ 3 & -6 & 9 & 3 \\ 2 & -3 & 8 & 9 \end{bmatrix}$$

$\frac{1}{3}r_2 \to r_2$

$$\begin{bmatrix} 0 & 1 & 2 & 5 \\ 1 & -2 & 3 & 1 \\ 2 & -3 & 8 & 9 \end{bmatrix}$$

$r_1 \Leftrightarrow r_2$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 5 \\ 2 & -3 & 8 & 9 \end{bmatrix}$$

$r_3 - 2r_1 \to r_3$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 7 \end{bmatrix}$$

$r_3 - r_2 \to r_3$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\frac{1}{2}r_3 \to r_3$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

REF

$r_1 - r_3 \to r_1$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$r_2 - 5r_3 \to r_2$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$r_1 + 2r_2 \to r_1$

$$\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

RREF