Homework 13 Solutions to Additional Problems:

1. Let $L : P_1 \to P_1$ be the linear transformation $L(at+b) = (a+4b)t + (a+b)$. Is L diagonalizable? If yes, find a basis S for P_1 such that the representation of L with respect to S is a diagonal matrix.

   We can check if L is diagonalizable by checking if a representation of L is diagonalizable. Let $T = \{t, 1\}$ be the standard basis for P_1. The representation of L with respect to T is $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$. To check if this matrix is diagonalizable, first find the eigenvalues. The eigenvalues are the roots of $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 1 & -4 \\ -1 & \lambda - 1 \end{bmatrix}\right) = (\lambda-1)(\lambda-1) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$. The eigenvalues are -1 and 3, each with multiplicity 1. As the eigenvalues are distinct (all multiplicity 1), A is diagonalizable and so is L.

   The basis S will be a basis consisting of eigenvectors of L. We start by finding the eigenvectors of A, then use these to find the eigenvectors of L. The eigenvectors associated with $\lambda = -1$ are the nonzero vectors in the null space of $-I - A = \begin{bmatrix} -2 & -4 \\ -1 & -2 \end{bmatrix}$. This matrix has RREF $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ so the eigenvectors are the nonzero multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. These are the T coordinates of the eigenvectors of L associated with -1, so the eigenvectors of L associated with -1 are the nonzero multiples of $-2t+1$. The eigenvectors associated with $\lambda = 3$ are the nonzero vectors in the null space of $3I - A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$. This matrix has RREF $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ so the eigenvectors are the nonzero multiples of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. These are the T coordinates of the eigenvectors of L associated with 3, so the eigenvectors of L associated with 3 are the nonzero multiples of $2t+1$. The set $S = \{-2t+1, 2t+1\}$ is a basis for P_1 and as these are both eigenvectors, the representation of L with respect to S will be diagonal. In particular, the representation is $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$.

2. For each matrix A, the characteristic polynomial of A is provided. Determine if A is diagonalizable.

   (a) $A = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 4 & -1 \\ 3 & 6 & -1 \end{bmatrix}$, $(\lambda-3)(\lambda-2)^2$

To check if A is diagonalizable, we check if the dimension of each eigenspace matches the multiplicity of the corresponding eigenvalue. This is automatically true for eigenvalues of multiplicity 1 so we only need to check the eigenvalues with multiplicity larger than 1. In this case, the eigenvalues are 3 with multiplicity 1 and 2 with multiplicity 2. We therefore need to check if the dimension of the eigenspace associated with $\lambda = 2$ is 2. The eigenspace associated with $\lambda = 2$ is the null space of $2\lambda - A = \begin{bmatrix} -2 & -4 & 2 \\ -1 & -2 & 1 \\ -3 & -6 & 3 \end{bmatrix}$. This has RREF $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. There are 2 columns without leading 1's, so the dimension of the eigenspace is 2 which matches the multiplicity so A is diagonalizable.

(b) $A = \begin{bmatrix} 3 & 0 & -2 & -1 \\ -1 & 2 & 5 & 4 \\ 6 & 0 & -5 & -3 \\ -6 & 0 & 4 & 2 \end{bmatrix}$, $\lambda(\lambda-1)(\lambda+1)(\lambda-2)$

The eigenvalues of A are $0, 1, -1, 2$ each with multiplicity 1. As the eigenvalues are distinct (each multiplicity 1), A is diagonalizable.

(c) $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 5 & -2 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}$, $(\lambda+1)(\lambda-1)^2(\lambda+2)$

Check if the eigenspace associated with $\lambda = 1$ has dimension 2. The eigenspace associated with $\lambda = 1$ is the null space of $I - A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -5 & 3 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix}$. This has RREF $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. There is only one column without a leading one so the eigenspace has dimension 1. This is less than the multiplicity so A is not diagonalizable.

3. Let A be a 3×3 matrix. Suppose that the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$ with associated eigenvectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$

respectively.

Find a formula for A^k and use this to find A and A^{100} (you do not need to simplify the entries of A^{100}).

The matrix A is diagonalizable, as it is 3×3 with 3 distinct eigenvalues. Start by finding a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$. The diagonal matrix will have the eigenvalues on the diagonal so $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. The invertible matrix P will have the eigenvectors as columns so $P = \begin{bmatrix} 1 & 2 & -2 \\ -1 & -2 & 3 \\ 1 & 3 & -3 \end{bmatrix}$. Note that the order in which you put the eigenvalues along the diagonal of D does not matter, but you must choose the same ordering for the columns of P. Solving the equation $D = P^{-1}AP$ for A we get that $A = PDP^{-1}$. Then $A^k = PD^kP^{-1}$.

If we start with $[P : I]$ and do row operations to get to $[I : P^{-1}]$, we find that $P^{-1} = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The powers of D are $D^k = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}$. Multiplying everything out we get $A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & -2 & 3 \\ 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} =$
$\begin{bmatrix} 3-2(3^k) & 2(2^k)-2(3^k) & -2+2(2^k) \\ -3+3(3^k) & -2(2^k)+3(3^k) & 2-2(2^k) \\ 3-3(3^k) & 3(2^k)-3(3^k) & -2+3(2^k) \end{bmatrix}$.

Plugging in $k = 1$, we get that $A = \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -3 & 4 \end{bmatrix}$. Plugging in $k = 100$,
$A^{100} = \begin{bmatrix} 3-2(3^{100}) & 2(2^{100})-2(3^{100}) & -2+2(2^{100}) \\ -3+3(3^{100}) & -2(2^{100})+3(3^{100}) & 2-2(2^{100}) \\ 3-3(3^{100}) & 3(2^{100})-3(3^{100}) & -2+3(2^{100}) \end{bmatrix}$.