Homework 12 Solutions to Additional Problems:

1. Let $L : P_1 \to P_1$ be the linear transformation $L(at+b) = (2a+7b)t + (2a-3b)$. Find all eigenvalues of L. For each eigenvalue, find all associated eigenvectors.

   There are few ways to do this problem. It can be done without representations, or using a representation with respect to any basis for P_1. Here we explain how to do this using the representation of L with respect to the standard basis for P_1.

   Let $S = \{t, 1\}$ and let A be the representation of L with respect to S. $L(t) = 2t+2, L(1) = 7t-3$ and the coordinates of these vectors with respect to S are $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -3 \end{bmatrix}$ respectively. Therefore $A = \begin{bmatrix} 2 & 7 \\ 2 & -3 \end{bmatrix}$. The eigenvalues of L will be the same as the eigenvalues of A. The eigenvalues of A are the roots of $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-2 & -7 \\ -2 & \lambda+3 \end{bmatrix}\right) = (\lambda-2)(\lambda+3) - 14 = \lambda^2 + \lambda - 20 = (\lambda+5)(\lambda-4)$. This is 0 when $\lambda = -5, 4$ so the eigenvalues of A and of L are -5 and 4.

   For $\lambda = -5$, the eigenvectors of A are the nonzero vectors in the null space of $-5I - A = \begin{bmatrix} -7 & -7 \\ -2 & -2 \end{bmatrix}$. This matrix has RREF $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. If we think of this as the coefficient matrix of a homogeneous linear system in x, y then y can be anything and $x = -y$. The null space is all vectors of the form $\begin{bmatrix} -y \\ y \end{bmatrix}$. A polynomial in P_1 is an eigenvector of L associated with -5 if and only if its coordinate with respect to S is an eigenvector of A associated with -5. Therefore the eigenvectors of L associated with -5 are all nonzero polynomials of the form $-yt + y = y(-t+1)$, or all nonzero scalar multiples of $-t+1$.

   For $\lambda = 4$, the eigenvectors of A are the nonzero vectors in the null space of $4I - A = \begin{bmatrix} 2 & -7 \\ -2 & 7 \end{bmatrix}$. This matrix has RREF $\begin{bmatrix} 1 & -7/2 \\ 0 & 0 \end{bmatrix}$. If we think of this as the coefficient matrix of a homogeneous linear system in x, y then y can be anything and $x = (7/2)y$. The null space is all vectors of the form $\begin{bmatrix} (7/2)y \\ y \end{bmatrix}$. The eigenvectors of L associated with 4 are all nonzero polynomials of the form $(7/2)yt + y = (y/2)(7t+2)$, or all nonzero scalar multiples of $7t+2$.

2. For each matrix A, find the eigenvalues of A. Find a basis for the eigenspace associated with each eigenvalue.

(a) $A = \begin{bmatrix} -1 & 0 & 0 \\ -4 & -5 & -8 \\ 4 & 4 & 7 \end{bmatrix}$

The characteristic polynomial of A is $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda+1 & 0 & 0 \\ 4 & \lambda+5 & 8 \\ -4 & -4 & \lambda-7 \end{bmatrix}\right) =$
$(\lambda+1)((\lambda+5)(\lambda-7)+32) = (\lambda+1)(\lambda^2-2\lambda-3) = (\lambda+1)(\lambda+1)(\lambda-3)$.
This is 0 when $\lambda = -1, 3$ so the eigenvalues are -1 and 3.

The eigenspace associated with $\lambda = -1$ is the null space of $-I - A = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{bmatrix}$. The RREF of this matrix is $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. If the we think of this as the coefficient matrix of a homogeneous linear system in the variables x, y, z then y, z can be anything as there are no leading ones in columns 2 and 3 and $x = -y - 2z$. The vectors in the eigenspace have the form $\begin{bmatrix} -y-2z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. This has basis $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}$.

The eigenspace associated with $\lambda = 3$ is the null space of $3I - A = \begin{bmatrix} 4 & 0 & 0 \\ 4 & 8 & 8 \\ -4 & -4 & -4 \end{bmatrix}$.

The RREF of this matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The solutions to this are that z can be anything, $y = -z$, and $x = 0$. The vectors in the eigenspace have the form $\begin{bmatrix} 0 \\ -z \\ z \end{bmatrix} = z\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. This has basis $\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$.

(b) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The characteristic polynomial of A is $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-1 & -2 & -3 & -4 \\ 0 & \lambda & -5 & -6 \\ 0 & 0 & \lambda-1 & -7 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}\right) =$
$\lambda(\lambda-1)^3$. This is 0 when $\lambda = 1, 0$ so the eigenvalues are 1 and 0.

The eigenspace associated with $\lambda = 1$ is the null space of $I - A = \begin{bmatrix} 0 & -2 & -3 & -4 \\ 0 & 1 & -5 & -6 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The RREF of this matrix is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Using variables x, y, z, w we see that $y = z = w = 0$ and x is anything. The vectors in the eigenspace have the form $\begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. This has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

The eigenspace associated with $\lambda = 0$ is the null space of $-A = \begin{bmatrix} -1 & -2 & -3 & -4 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

The RREF of this matrix is $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solutions to this are that $z = w = 0$, y can be anything, and $x = -2y$. The vectors in the eigenspace have the form $\begin{bmatrix} -2y \\ y \\ 0 \\ 0 \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. This has basis $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

3. Let A be an $n \times n$ matrix. Prove that A and A^T have the same eigenvalues. Do they have the same eigenvectors?

The eigenvalues of A are the roots of the characteristic polynomial $\det(\lambda I - A)$. The eigenvalues of A^T are the roots of $\det(\lambda I - A^T)$. Note that $(\lambda I - A)^T = (\lambda I)^T - A^T = \lambda I - A^T$ since λI is symmetric. Also, the determinant of a matrix is the same as the determinant of the transpose, so $\det(\lambda I - A) = \det((\lambda I - A)^T) = \det(\lambda I - A^T)$. We have proved that A and A^T have the same characteristic polynomial so they have the same eigenvalues.

The eigenvectors are not the same. For example, the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The transpose, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ does not have eigenvector

$\begin{bmatrix}1\\0\end{bmatrix}$ because $\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$ which is not a multiple of $\begin{bmatrix}1\\0\end{bmatrix}$.

4. Let λ be an eigenvalue of an $n \times n$ matrix A with associated eigenvector $\mathbf{v}$. Prove one of the following statements (you do not need to prove all 3).

   (a) $\mathbf{v}$ is also an eigenvector of A^2 with associated eigenvalue λ^2.

   A has eigenvalue λ with associated eigenvector $\mathbf{v}$ so $A\mathbf{v} = \lambda\mathbf{v}$. Then $A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v})) = \lambda^2\mathbf{v}$. We have shown that $A^2\mathbf{v} = \lambda^2\mathbf{v}$ so $\mathbf{v}$ is an eigenvector of A^2 with associated eigenvalue λ^2.

   (b) $\mathbf{v}$ is also an eigenvector of A^{-1} with associated eigenvalue $1/\lambda$ (assuming A is invertible).

   Assume A is invertible. In one of the book problems, you showed that if A is invertible then 0 is not an eigenvalue of A so $\lambda \neq 0$. A has eigenvalue λ with associated eigenvector $\mathbf{v}$ so $A\mathbf{v} = \lambda\mathbf{v}$. If we multiply both sides of this equation by A^{-1} on the left, we get $A^{-1}A\mathbf{v} = A^{-1}(\lambda\mathbf{v})$. This simplifies to $\mathbf{v} = \lambda A^{-1}\mathbf{v}$. As $\lambda \neq 0$, we divide both sides by λ to get $A^{-1}\mathbf{v} = (1/\lambda)\mathbf{v}$ so $\mathbf{v}$ is an eigenvector of A^{-1} with associated eigenvalue $1/\lambda$.

   (c) $\mathbf{v}$ is also an eigenvector of $A + rI$ with associated eigenvalue $\lambda + r$.

   A has eigenvalue λ with associated eigenvector $\mathbf{v}$ so $A\mathbf{v} = \lambda\mathbf{v}$. Then $(A + rI)\mathbf{v} = A\mathbf{v} + rI\mathbf{v} = \lambda\mathbf{v} + r\mathbf{v} = (\lambda + r)\mathbf{v}$. We have shown that $(A + rI)\mathbf{v} = (\lambda + r)\mathbf{v}$ so $\mathbf{v}$ is an eigenvector of $A + rI$ with associated eigenvalue $\lambda + r$.