Homework 11 Solutions to Additional Problems:

1. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ y-x \end{bmatrix}$.

   Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\}$.
   Let A be the standard matrix representing L and let B be the representation of L with respect to T. Let P be the transition matrix from T to S and let Q be the transition matrix from S to T.

   (a) Find A.

   This is the matrix whose columns are $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Note that the standard matrix representing L is the same as the representation of L with respect to S.

   (b) Find P and Q.

   To find P, take the vectors in T and find their coordinates with respect to S. As S is the standard basis, taking coordinates will not change the vector so $P = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$.

   To find Q, either take the vectors in S and find their T coordinates or use that $Q = P^{-1}$. Using the inverse formula for 2×2 matrices, we get that $Q = P^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$.

   (c) Using the methods of Section 6.5, find a formula for B in terms of A, P, Q and use this to compute B.

   As A is the representation of L with respect to S, we will have that $B = QAP = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}$.

   (d) Compute B using the methods of Section 6.3.

   To find B this way, plug the vectors from T into L and then take coordinates with respect to T. $L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $L\left(\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The coordinate vector of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ with respect to T is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. To find the coordinate vector of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

with respect to T, we solve the system $\begin{bmatrix}1\\-1\end{bmatrix} = x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}-1\\-2\end{bmatrix}$. The solution is $x = 3, y = 2$ so the coordinate vector is $\begin{bmatrix}3\\2\end{bmatrix}$. The two coordinate vectors we found are the columns of B so $B = \begin{bmatrix}0 & 3\\0 & 2\end{bmatrix}$. This matches what we found in the previous part.

2. Let $L : P_2 \to \mathbb{R}_2$ be the linear transformation $L(at^2 + bt + c) = \begin{bmatrix}a - b & b + 2c\end{bmatrix}$. Let S, S', T, T' be the following bases for P_2 and $\mathbb{R}_2$.

$$S = \{t^2, t, 1\}, S' = \{t^2 + 2t - 1, 3t + 5, 2t^2 + t - 4\}$$

$$T = \left\{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\end{bmatrix}\right\}, T' = \left\{\begin{bmatrix}1 & -1\end{bmatrix}, \begin{bmatrix}2 & 0\end{bmatrix}\right\}$$

Let A be the matrix representing L with respect to S and T and let B be the matrix representing L with respect to S' and T'.

(a) Find A.

Plug the vectors from S into L then take T coordinates. $L(t^2) = \begin{bmatrix}1 & 0\end{bmatrix}, L(t) = \begin{bmatrix}-1 & 1\end{bmatrix}, L(1) = \begin{bmatrix}0 & 2\end{bmatrix}$. The coordinates with respect to T are $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}$ respectively, so $A = \begin{bmatrix}1 & -1 & 0\\0 & 1 & 2\end{bmatrix}$.

(b) Let C be the transition matrix from S to S' and let D be the transition matrix from T to T'. Fill in the blanks with C, C^{-1}, D, or D^{-1}:

$$B = DAC^{-1}$$

(c) Compute the transition matrices you used in the previous part and use that formula to compute B.

We need to compute D and C^{-1}. D is the transition matrix from T to T'. The transition matrix from T' to T is easier to compute, so we instead compute this and then take its inverse to get D. To find the transition matrix from T' to T, we take each vector in T' and find its coordinate vector with respect to T. The coordinate vectors are $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\0\end{bmatrix}$ so $D^{-1} = \begin{bmatrix}1 & 2\\-1 & 0\end{bmatrix}$. Using the 2×2 inverse formula, we get that $D = \begin{bmatrix}0 & -1\\1/2 & 1/2\end{bmatrix}$.

C^{-1} is the transition matrix from S' to S so we take the vectors in S' and find their coordinates with respect to S. The coordinate vectors are $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$ so $C^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ -1 & 5 & -4 \end{bmatrix}$.

Then $B = DAC^{-1} = \begin{bmatrix} 0 & -1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ -1 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 0 & -13 & 7 \\ -1/2 & 5 & -3 \end{bmatrix}$.

(d) Compute B using the methods of Section 6.3.

Take the vectors in S' and plug them into L then find the T' coordinates. $L(t^2 + 2t - 1) = \begin{bmatrix} -1 & 0 \end{bmatrix}, L(3t + 5) = \begin{bmatrix} -3 & 13 \end{bmatrix}, L(2t^2 + t - 4) = \begin{bmatrix} 1 & -7 \end{bmatrix}$. To find the coordinates of $\begin{bmatrix} -1 & 0 \end{bmatrix}$ with respect to T', we solve the system $\begin{bmatrix} -1 & 0 \end{bmatrix} = x \begin{bmatrix} 1 & -1 \end{bmatrix} + y \begin{bmatrix} 2 & 0 \end{bmatrix}$. The solution is $x = 0, y = -1/2$ so the coordinate vector is $\begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$. To find the coordinates of $\begin{bmatrix} -3 & 13 \end{bmatrix}$, we solve the system $\begin{bmatrix} -3 & 13 \end{bmatrix} = x \begin{bmatrix} 1 & -1 \end{bmatrix} + y \begin{bmatrix} 2 & 0 \end{bmatrix}$. The solution is $x = -13, y = 5$ so the coordinate vector is $\begin{bmatrix} -13 \\ 5 \end{bmatrix}$.To find the coordinates of $\begin{bmatrix} 1 & -7 \end{bmatrix}$, we solve the system $\begin{bmatrix} 1 & -7 \end{bmatrix} = x \begin{bmatrix} 1 & -1 \end{bmatrix} + y \begin{bmatrix} 2 & 0 \end{bmatrix}$. The solution is $x = 7, y = -3$ so the coordinate vector is $\begin{bmatrix} 7 \\ -3 \end{bmatrix}$. These coordinates are the columns of B so $B = \begin{bmatrix} 0 & -13 & 7 \\ -1/2 & 5 & -3 \end{bmatrix}$.

3. Suppose A and B are similar matrices. Prove that $A + rI$ and $B + rI$ are similar matrices for any real number r.

A and B are similar so $B = P^{-1}AP$ for some invertible matrix P. Then

$$P^{-1}(A + rI)P = P^{-1}AP + P^{-1}(rI)P = B + r(P^{-1}IP) = B + rI$$

so the matrices $A + rI$ and $B + rI$ are similar.