Homework 10 Solutions to Additional Problems:

1. Let $L : \mathbb{R}^2 \to P_1$ be the linear transformation $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (2a+5b)t + (a+3b)$. Show that L is invertible and find L^{-1}.

We will give two possible methods for solving this problem.

Method 1: Without representations

To show L is invertible, show that it is one-to-one and onto. The kernel of L is all 2-vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ with $2a+5b=0$ and $a+3b=0$. The only solution here is $a=b=0$ so $\ker L = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ and therefore L is one-to-one. The range of L is all polynomials of the form $(2a+5b)t+(a+3b) = a(2t+1)+b(5t+3)$ so range $L = \text{span}\{2t+1, 5t+3\}$. These two vectors are linearly independent so the range has dimension 2 and is therefore all of P_1. This shows L is onto.

To find $L^{-1} : P_2 \to \mathbb{R}^2$, we can look at what L does to the standard basis for $\mathbb{R}^2$. We use that $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2t+1$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 5t+3$ to get $L^{-1}(2t+1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $L^{-1}(5t+3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Solving the equation $at+b = x(2t+1)+y(5t+3)$ for x and y in terms of a and b, we get $x = 3a-5b, y = -a+2b$. Thus $at+b = (3a-5b)(2t+1)+(-a+2b)(5t+3)$. Then $L^{-1}(at+b) = L^{-1}((3a-5b)(2t+1)+(-a+2b)(5t+3)) = (3a-5b)L^{-1}(2t+1)+(-a+2b)L^{-1}(5t+3) = (3a-5b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-a+2b)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3a-5b \\ -a+2b \end{bmatrix}$.

Method 2: Using representations

Let $S = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and $T = \{t, 1\}$. Let A be the representation of L with respect to S and T. Then $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2t+1$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 5t+3$. The coordinate vectors of these vectors with respect to T are $[2t+1]_T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $[5t+3]_T = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Therefore $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$. As $\det(A) = 1$, A is an invertible matrix so L is an invertible linear transformation.

The inverse $L^{-1} : P_1 \to \mathbb{R}^2$ has representation A^{-1} with respect to T and S. Using the 2×2 formula for inverses, $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$. For any $at + b$ in P_1, we get that $[L^{-1}(at+b)]_S = A^{-1}[at+b]_T$. T is the standard basis for P_1 so $[at+b]_T = \begin{bmatrix} a \\ b \end{bmatrix}$. Plugging in the matrix A^{-1} we get that $[L^{-1}(at + b)]_S = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a - 5b \\ -a + 2b \end{bmatrix}$. S is the standard basis for $\mathbb{R}^2$ so taking the coordinate of a 2-vector with respect to S does not change the vector so $L^{-1}(at + b) = [L^{-1}(at + b)]_S = \begin{bmatrix} 3a - 5b \\ -a + 2b \end{bmatrix}$.

2. Let $L : P_2 \to \mathbb{R}^4$ be the linear transformation $L(at^2 + bt + c) = \begin{bmatrix} a+b+c \\ a-b+c \\ 2b \\ b-a-c \end{bmatrix}$.

   (a) Find a basis for $\ker L$.

   The kernel of L is the set of polynomials $at^2 + bt + c$ such that $a + b + c = 0, a - b + c = 0, 2b = 0, b - a - c = 0$. The solutions to this homogeneous linear system are that a is anything, $b = 0$, and $c = -a$. Thus $\ker L = \{at^2 - a\} = \text{span}\{t^2 - 1\}$. This has basis $\{t^2 - 1\}$.

   (b) Find a basis for range$L$.

   The range of L is all 4-vectors of the form $\begin{bmatrix} a+b+c \\ a-b+c \\ 2b \\ b-a-c \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Thus range $L = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$. This has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

   (c) Find the representation of L with respect to S and T where S and T are the following bases for P_2 and $\mathbb{R}^4$ respectively. $S = \{t^2 + 2t - 1, 3t + 5, 2t^2 + t - 4\}$,

$$T = \left\{ \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$$

Start by plugging the vectors from S into L. $L(t^2+2t-1) = \begin{bmatrix}2\\-2\\4\\2\end{bmatrix}$, $L(3t+5) = \begin{bmatrix}8\\2\\6\\-2\end{bmatrix}$, $L(2t^2+t-4) = \begin{bmatrix}-1\\-3\\2\\3\end{bmatrix}$. Then take the coordinate vectors of each of these vectors with respect to T. For the first vecter, $\begin{bmatrix}2\\-2\\4\\2\end{bmatrix}$, this means solving the linear system $\begin{bmatrix}2\\-2\\4\\2\end{bmatrix} = x\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + y\begin{bmatrix}1\\0\\0\\2\end{bmatrix} + z\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + w\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$. This has solution $x = -6, y = 4, z = 4, w = -4$ so the coordinate vector is $\begin{bmatrix}-6\\4\\4\\-4\end{bmatrix}$. Similarly, the coordinate vectors for the other two vectors are $\begin{bmatrix}-4\\6\\6\\-16\end{bmatrix}$ and $\begin{bmatrix}-5\\2\\2\\2\end{bmatrix}$ respectively. Thus the representation is $\begin{bmatrix}-6 & -4 & -5\\4 & 6 & 2\\4 & 6 & 2\\-4 & -16 & 2\end{bmatrix}$.

3. Let $L : M_{nn} \to \mathbb{R}$ be the linear transformation $L(A) = a_{11} + a_{22} + ... + a_{nn}$ where a_{ij} is the i, j-th entry of A. Find $\dim \ker L$ and $\dim \text{range} L$ (your answers may depend on n). Is L one-to-one? Onto?

   $\dim \ker L = n^2 - 1$ and $\dim \text{range } L = 1$. In this case, it's easier to find the dimension of the range. The range is a subspace of $\mathbb{R}$ so it is either $\{0\}$ or $\mathbb{R}$. As it is possible to get any real number as the result of the function L, the range of L is $\mathbb{R}$ so it has dimension 1. Then $\dim \ker L + \dim \text{range } L = \dim M_{nn}$ so $\dim \ker L = n^2 - 1$.

L is always onto, but it only one-to-one if $n = 1$.

4. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and let $T = \left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$. These are bases for $\mathbb{R}^3$ and M_{22} respectively. Let $L : \mathbb{R}^3 \to M_{22}$ be a linear transformation such that the representation of L with respect to S and T is $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \\ 1 & 4 & -1 \end{bmatrix}$. Find $L\left(\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \right)$.

Let $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$. We will use that $[L(\mathbf{v})]_T = A[\mathbf{v}]_S$. To find $[\mathbf{v}]_S$, we need to solve the linear system $\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. This has solution $x = 5, y = -1, z = 1$ so $[\mathbf{v}]_S = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$. Then $[L(\mathbf{v})]_T = A[\mathbf{v}]_S = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \\ 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}$. Therefore $L(\mathbf{v}) = 2 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + (-2) \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 0 & 0 \end{bmatrix}$.