Solutions to Final Exam

1. Let A be a 3×5 matrix. Let $\mathbf{b}$ be a nonzero 5-vector. Assume that the nullity of A is 2. (14 pts)

   (a) What is the rank of A? 3

   (b) Are the rows of A linearly independent? yes

   (c) Are the columns of A linearly independent? no

   (d) How many solutions does the linear system $A\mathbf{x} = \mathbf{0}$ have? infinite

   (e) How many solutions does the linear system $A\mathbf{x} = \mathbf{b}$ have? (list all possible numbers of solutions) infinite

   (f) Let $\mathbf{v}$ be a solution to $A\mathbf{x} = \mathbf{0}$ and $\mathbf{w}$ be a solution to $A\mathbf{x} = \mathbf{b}$. Find all scalars r and s such that $r\mathbf{v} + s\mathbf{w}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

   From the given information, $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{b}$. Then $A(r\mathbf{v} + s\mathbf{w}) = r(A\mathbf{v}) + s(A\mathbf{w}) = r\mathbf{0} + s\mathbf{b} = s\mathbf{b}$. As $\mathbf{b} \neq \mathbf{0}$, this equals $\mathbf{b}$ if and only if $s = 1$, so we get that r can be anything and $s = 1$.

2. Let B be a 4×4 matrix such that $B^{-1} = \frac{1}{2}B^T$.

   (a) Find all possible values of $\det(B)$. (6 pts)

   Taking determinants of both sides of $B^{-1} = \frac{1}{2}B^T$ gives us $\det(B^{-1}) = \det(\frac{1}{2}B^T)$. Using determinant properties to simplify both sides, we get $1/\det(B) = \left(\frac{1}{2}\right)^4 \det(B)$ so $\det(B)^2 = 16$ and $\det(B) = \pm 4$.

   (b) What is the RREF of B? (3 pts)

   B is an invertible 4×4 matrix so the RREF of B is the 4×4 identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 5 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 6 & -2 \\ 0 & 0 & -2 & 5 \end{bmatrix}$. One of the eigenvalues of A is 1.

(a) Find the characteristic polynomial of A and all eigenvalues of A. (8 pts)

$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-5 & 0 & -4 & 0 \\ 0 & \lambda-1 & 0 & 0 \\ -4 & 0 & \lambda-6 & 2 \\ 0 & 0 & 2 & \lambda-5 \end{bmatrix}\right)$. Using cofactor expansion on column 2, this is $(\lambda-1)\det\left(\begin{bmatrix} \lambda-5 & -4 & 0 \\ -4 & \lambda-6 & 2 \\ 0 & 2 & \lambda-5 \end{bmatrix}\right) = (\lambda-1)[(\lambda-5)(\lambda-6)(\lambda-5) - 4(\lambda-5) - 16(\lambda-5)] = (\lambda-1)(\lambda-5)[(\lambda-6)(\lambda-5)-20] = (\lambda-1)(\lambda-5)(\lambda^2-11\lambda+10) = (\lambda-1)(\lambda-5)(\lambda-1)(\lambda-10)$. The eigenvalues are 1 (with multiplicity 2), 5 (with multiplicity 1), and 10 (with multiplicity 1).

(b) Find a basis for the eigenspace associated with the eigenvalue 1. (8 pts)

The eigenspace associated with 1 is the same as the null space of $I - A = \begin{bmatrix} -4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & -5 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$. The RREF of this matrix is $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. If this is the coefficient matrix of a homogeneous linear system in x, y, z, w, then columns 2 and 4 do not have leading 1's so the variables y, w can be anything and $x = -2w, y = 2w$. The eigenspace is therefore all vectors of the form $\begin{bmatrix} -2w \\ y \\ 2w \\ w \end{bmatrix} = w\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. The set $\left\{\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ spans the eigenspace and is linearly independent, so it is a basis for the eigenspace.

(c) Is A diagonalizable? Why or why not? (4 pts)

Yes. The dimension of the eigenspaces match the multiplicities of the eigenvectors (this is always true for the multiplicity 1 eigenvalues so we only need to check it for the eigenvalue 1). Also, this is a symmetric matrix and symmetric matrices are always diagonalizable.

4. Let S be the set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(a) Determine if S is orthogonal, orthonormal, or neither. Explain. (4 pts)

S is orthogonal. It is orthogonal because the dot product of any 2 distinct vectors in S is 0 (there are 3 pairs to check). It is not orthonormal however as the vectors are not length 1. The lengths are 2, $\sqrt{2}$, and $\sqrt{6}$ respectively.

(b) Is S a linearly independent set? Why or why not? (4 pts)

Yes. An orthogonal set of nonzero vectors is always linearly independent.

(c) Find an orthonormal basis for span S. (4 pts)

As S is linearly independent, S is a basis for span S. It is also already orthogonal, so to make it orthonormal we just need to divide each vector by its length. The resulting orthonormal basis we get is:

$$\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$

5. Let $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \right\}$.

Let c be a constant. For what value or values of c is the matrix $\begin{bmatrix} c^2 & -5c \\ c^2-4 & -6 \end{bmatrix}$ in the span of S? (8 pts)

The vectors in S all have 0's in the bottom left, so for $\begin{bmatrix} c^2 & -5c \\ c^2-4 & -6 \end{bmatrix}$ to be the span of S, we would need $c^2 - 4 = 0$ so $c = \pm 2$ are the only possibilities. We check each case. The matrices are $\begin{bmatrix} 4 & -10 \\ 0 & -6 \end{bmatrix}$ when $c = 2$ and $\begin{bmatrix} 4 & 10 \\ 0 & -6 \end{bmatrix}$ when $c = -2$. Note that the set S is not linearly independent as the third vector is the sum of the first two, so we can delete the third without changing the span. For $c = 2$, we check if there are x, y such that $\begin{bmatrix} 4 & -10 \\ 0 & -6 \end{bmatrix} = x \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

This happens when $x = 4, y = -18$ so when $c = 2$ the matrix is in the span. For $c = -2$, we check if there are x, y such that $\begin{bmatrix} 4 & 10 \\ 0 & -6 \end{bmatrix} = x \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. This gives us the linear system $x = 4, 2x + y = 10, 3x + y = -6$, which has no solutions. So when $c = -2$, the matrix is not in the span. Therefore the only constant c for which $\begin{bmatrix} c^2 & -5c \\ c^2 - 4 & -6 \end{bmatrix}$ is in the span of S is $c = 2$.

6. Let W be the set of polynomials $p(t)$ in P_3 with the property that $p(1) = p(-1)$. W is a subspace of P_3. Find a basis for W and $\dim W$. (8 pts)

   The polynomials in P_3 look like $p(t) = at^3 + bt^2 + ct + d$, so $p(1) = a + b + c + d$ and $p(-1) = -a + b - c + d$. To be in W, $p(t) = at^3 + bt^2 + ct + d$ must have $a+b+c+d = -a+b-c+d$ which simplifies to $c = -a$. Therefore the polynomials in W are the polynomials of the form $at^3 + bt^2 - at + d = a(t^3 - t) + bt^2 + d(1)$. The set $\{t^3 - t, t^2, 1\}$ spans W and is linearly independent so it is a basis for W and $\dim W = 3$.

7. Let $L : U \to V$ be a linear transformation where $\dim U = 3$ and $\dim V = 4$.
   Let $R = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ and $S = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be bases for U.
   Let $T = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ be a basis for V.
   Let A be the representation of L with respect to R and T.
   Let B be the representation of L with respect to S and T.
   Let C be the transition matrix from R to S.
   Answer the following multiple choice questions (circle the best answer).

Question 1: The kernel of L is a subspace of which space? (2 pts)

(a) U

Question 2: The range of L is a subspace of which space? (2 pts)

(b) V

Question 3: From the dimensions of U and V, we can tell that L is ________. (2 pts)

(d) not onto

Question 4: If $\dim \ker L = 1$, what is $\dim \operatorname{range} L$? (2 pts)

(c) 2

Question 5: Which of the following is the first column of A? (2 pts)

(d) $[L(\mathbf{u_1})]_T$

Question 6: Which of the following is the first column of C? (2 pts)

(b) $[\mathbf{u_1}]_S$

Question 7: Which of the following is equal to B? (3 pts)

(d) AC^{-1}

8. Let $A = \begin{bmatrix} 9 & -2 & -4 \\ 8 & -1 & -4 \\ 8 & -2 & -3 \end{bmatrix}$ and let $P = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 2 & 1 & 7 \end{bmatrix}$.

P is an invertible matrix with inverse $P^{-1} = \begin{bmatrix} -9 & 4 & 5 \\ 4 & -1 & -2 \\ 2 & -1 & -1 \end{bmatrix}$.

(a) Prove that the columns of P are eigenvectors of A and find their associated eigenvalues. (6 pts)

$\begin{bmatrix} 9 & -2 & -4 \\ 8 & -1 & -4 \\ 8 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ so the first column is an eigenvector with associated eigenvalue 1. $\begin{bmatrix} 9 & -2 & -4 \\ 8 & -1 & -4 \\ 8 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so the second column is an eigenvector with associated eigenvalue 3. $\begin{bmatrix} 9 & -2 & -4 \\ 8 & -1 & -4 \\ 8 & -2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$ so the third column is an eigenvector with associated eigenvalue 1.

(b) Find A^{80}.
Note: Your answer should be a single matrix, but you do not need to simplify the entries. (8 pts)

A is diagonalizable as it has three linearly independent eigenvectors. The columns of P are the eigenvectors, so $P^{-1}AP = D$ where D is the diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Solving for A we get that $A = PDP^{-1}$ and $A^{80} = PD^{80}P^{-1}$. Note that D^{80} is the diagonal matrix whose diagonal entries are $1^{80} = 1, 3^{80}, 1^{80} = 1$. We can thus multiply everything out to get A^{80}.

$$A^{80} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^{80} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -9 & 4 & 5 \\ 4 & -1 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} -9 & 4 & 5 \\ 4(3^{80}) & -(3^{80}) & -2(3^{80}) \\ 2 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 + 4(3^{80}) & 1 - (3^{80}) & 2 - 2(3^{80}) \\ -4 + 4(3^{80}) & 2 - (3^{80}) & 2 - 2(3^{80}) \\ -4 + 4(3^{80}) & 1 - (3^{80}) & 3 - 2(3^{80}) \end{bmatrix}$$

Bonus: Find the characteristic polynomial of the 20×20 all ones matrix. (5 pts)

Let J_n denote the $n \times n$ all ones matrix. A good way to approach this problem is to think about the cases for small n and look for patterns. The matrix J_n has nullity $n-1$, so 0 is an eigenvalue and the eigenspace associated with 0 has dimension $n-1$. Also, J_n has the all ones n-vector as an eigenvector with associated eigenvalue n. Thus J_{20} has 0 as an eigenvalue with multiplicity at least 19 and eigenvalue 20 with multiplicity at least 1. The multiplicities add up to 20 so the multiplicities of 0 and 20 must be 19 and 1 respectively, so the characteristic polynomial is $\lambda^{19}(\lambda - 20)$. In general, J_n has characteristic polynomial $\lambda^{n-1}(\lambda - n)$.