Math 3333
Spring 2015
Final Exam

Name:____

Problem	**Points**
Problem 1 (14pts)	
Problem 2 (9pts)	
Problem 3 (20pts)	
Problem 4 (12pts)	
Problem 5 (8pts)	
Problem 6 (8pts)	
Problem 7 (15pts)	
Problem 8 (14pts)	
Bonus (5pts)	
Total	

1. Let A be a 3×5 matrix. Let $\mathbf{b}$ be a nonzero 5-vector. Assume that the nullity of A is 2. (14 pts)

   (a) What is the rank of A? __________

   (b) Are the rows of A linearly independent? __________

   (c) Are the columns of A linearly independent? __________

   (d) How many solutions does the linear system $A\mathbf{x} = \mathbf{0}$ have? __________

   (e) How many solutions does the linear system $A\mathbf{x} = \mathbf{b}$ have?
   (list all possible numbers of solutions) __________

   (f) Let $\mathbf{v}$ be a solution to $A\mathbf{x} = \mathbf{0}$ and $\mathbf{w}$ be a solution to $A\mathbf{x} = \mathbf{b}$.
   Find all scalars r and s such that $r\mathbf{v} + s\mathbf{w}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

2. Let B be a 4×4 matrix such that $B^{-1} = \frac{1}{2}B^T$.

   (a) Find all possible values of $\det(B)$. (6 pts)

   (b) What is the RREF of B? (3 pts)

3. Let $A = \begin{bmatrix} 5 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 6 & -2 \\ 0 & 0 & -2 & 5 \end{bmatrix}$. One of the eigenvalues of A is 1.

(a) Find the characteristic polynomial of A and all eigenvalues of A. (8 pts)

(b) Find a basis for the eigenspace associated with the eigenvalue 1. (8 pts)

(c) Is A diagonalizable? Why or why not? (4 pts)

4. Let S be the set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(a) Determine if S is orthogonal, orthonormal, or neither. Explain. (4 pts)

(b) Is S a linearly independent set? Why or why not? (4 pts)

(c) Find an orthonormal basis for span S. (4 pts)

5. Let $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \right\}$.

Let c be a constant. For what value or values of c is the matrix $\begin{bmatrix} c^2 & -5c \\ c^2 - 4 & -6 \end{bmatrix}$ in the span of S? (8 pts)

6. Let W be the set of polynomials $p(t)$ in P_3 with the property that $p(1) = p(-1)$. W is a subspace of P_3. Find a basis for W and $\dim W$. (8 pts)

7. Let $L : U \to V$ be a linear transformation where $\dim U = 3$ and $\dim V = 4$.
Let $R = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ and $S = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be bases for U.
Let $T = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ be a basis for V.
Let A be the representation of L with respect to R and T.
Let B be the representation of L with respect to S and T.
Let C be the transition matrix from R to S.
Answer the following multiple choice questions (circle the best answer).

Question 1: The kernel of L is a subspace of which space? (2 pts)

(a) U (b) V (c) $\mathbb{R}^3$ (d) $\mathbb{R}^4$

Question 2: The range of L is a subspace of which space? (2 pts)

(a) U (b) V (c) $\mathbb{R}^3$ (d) $\mathbb{R}^4$

Question 3: From the dimensions of U and V, we can tell that L is ________. (2 pts)

(a) one-to-one
(b) onto
(c) not one-to-one
(d) not onto
(e) both one-to-one and onto
(f) neither one-to-one nor onto

Question 4: If $\dim \ker L = 1$, what is $\dim \operatorname{range} L$? (2 pts)

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Question 5: Which of the following is the first column of A? (2 pts)

(a) $[\mathbf{u_1}]_R$
(b) $[\mathbf{u_1}]_T$
(c) $[L(\mathbf{u_1})]_R$
(d) $[L(\mathbf{u_1})]_T$
(e) $[\mathbf{v_1}]_R$
(f) $[\mathbf{v_1}]_T$
(g) $[L(\mathbf{v_1})]_R$
(h) $[L(\mathbf{v_1})]_T$

Question 6: Which of the following is the first column of C? (2 pts)

(a) $[\mathbf{u_1}]_R$
(b) $[\mathbf{u_1}]_S$
(c) $[L(\mathbf{u_1})]_R$
(d) $[L(\mathbf{u_1})]_S$
(e) $[\mathbf{w_1}]_R$
(f) $[\mathbf{w_1}]_S$
(g) $[L(\mathbf{w_1})]_R$
(h) $[L(\mathbf{w_1})]_S$

Question 7: Which of the following is equal to B? (3 pts)

(a) CA
(b) AC
(c) $C^{-1}A$
(d) AC^{-1}
(e) $C^{-1}AC$
(f) CAC^{-1}

8. Let $A = \begin{bmatrix} 9 & -2 & -4 \\ 8 & -1 & -4 \\ 8 & -2 & -3 \end{bmatrix}$ and let $P = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 2 & 1 & 7 \end{bmatrix}$.

P is an invertible matrix with inverse $P^{-1} = \begin{bmatrix} -9 & 4 & 5 \\ 4 & -1 & -2 \\ 2 & -1 & -1 \end{bmatrix}$.

(a) Prove that the columns of P are eigenvectors of A and find their associated eigenvalues. (6 pts)

(b) Find A^{80}.
Note: Your answer should be a single matrix, but you do not need to simplify the entries. (8 pts)

Bonus: Find the characteristic polynomial of the following 20×20 matrix. (5 pts)
You must show work or explain your answer to get credit.

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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\end{bmatrix}$$