# Exam 3 Solutions

1. Let $S$ be the following orthonormal basis for $\mathbb{R}^3$.

$$S = \left\{ \frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}, \frac{1}{3}\begin{bmatrix}-2\\2\\-1\end{bmatrix}, \frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix} \right\}$$

If $\mathbf{v} = \begin{bmatrix}2\\1\\4\end{bmatrix}$, find $[\mathbf{v}]_S$. (10 pts)

We want to find $x, y, z$ with
$\begin{bmatrix}2\\1\\4\end{bmatrix} = x\left(\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}\right) + y\left(\frac{1}{3}\begin{bmatrix}-2\\2\\-1\end{bmatrix}\right) + z\left(\frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix}\right)$. As $S$ is orthonormal,
$x = \begin{bmatrix}2\\1\\4\end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix} = \frac{1}{3}(2+2+8) = 4$, $y = \begin{bmatrix}2\\1\\4\end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix}-2\\2\\-1\end{bmatrix} = \frac{1}{3}(-4+2+-4) = -2$,
$z = \begin{bmatrix}2\\1\\4\end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix} = \frac{1}{3}(4+1-8) = -1$. Then $[\mathbf{v}]_S = \begin{bmatrix}4\\-2\\-1\end{bmatrix}$.

2. Let $V$ be a 2-dimensional subspace of $\mathbb{R}^n$ with basis $T = \{\mathbf{v_1}, \mathbf{v_2}\}$.
Suppose that $\|\mathbf{v_1}\| = \sqrt{3}$, $\|\mathbf{v_2}\| = \sqrt{5}$, and $\mathbf{v_1} \cdot \mathbf{v_2} = 2$.
Let $\mathbf{u} = \mathbf{v_1} - 2\mathbf{v_2}$. Find $\|\mathbf{u}\|$. (12 pts)

The length of $\mathbf{u}$ is $\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$. This formula also tells us that $\mathbf{v_1}\cdot\mathbf{v_1} = 3$ and $\mathbf{v_2}\cdot\mathbf{v_2} = 5$. Using the properties of dot product, we get that
$\mathbf{u}\cdot\mathbf{u} = (\mathbf{v_1} - 2\mathbf{v_2})\cdot(\mathbf{v_1} - 2\mathbf{v_2}) = (\mathbf{v_1}\cdot\mathbf{v_1}) - 2(\mathbf{v_1}\cdot\mathbf{v_2}) - 2(\mathbf{v_2}\cdot\mathbf{v_1}) + 4(\mathbf{v_2}\cdot\mathbf{v_2}) = 3 - 2(2) - 2(2) + 5(4) = 15$. Therefore $\|\mathbf{u}\| = \sqrt{15}$.

3. Let $L : \mathbb{R}^2 \to \mathbb{R}$ be the function $L\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \sqrt{x^2+y^2}$.
Is $L$ a linear transformation? Why or why not? (10 pts)

This is not a linear transformation. It fails both of the linear transformation properties. You only need to show that one fails, but here we will show both.

Property 1: $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$
If $\mathbf{u} = \begin{bmatrix}x\\y\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}x'\\y'\end{bmatrix}$, then $L(\mathbf{u}+\mathbf{v}) = \sqrt{(x+x')^2 + (y+y')^2}$ and $L(\mathbf{u}) + L(\mathbf{v}) = \sqrt{x^2+y^2} + \sqrt{(x')^2+(y')^2}$. These are not equal. For example if $\mathbf{u} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}0\\1\end{bmatrix}$ then $L(\mathbf{u}+\mathbf{v}) = \sqrt{2}$ and $L(\mathbf{u}) + L(\mathbf{v}) = 2$.

Property 2: $L(r\mathbf{v}) = rL(\mathbf{v})$

If $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $L(r\mathbf{v}) = \sqrt{(rx)^2 + (ry)^2} = |r|\sqrt{x^2 + y^2}$ and $rL(\mathbf{v}) = r\sqrt{x^2 + y^2}$. These are not equal when $r < 0$. For example, if $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $r = -1$ then $L(r\mathbf{v}) = 1$ and $rL(\mathbf{v}) = -1$.

Note that $L$ is the length function, so you can also think about these properties visually by representing a vector in $\mathbb{R}^2$ with a directed line segment on the $xy$-plane. The two properties are the same as checking if $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ and $\|r\mathbf{v}\| = r\|\mathbf{v}\|$. The first one only holds when the vectors point in the same direction and the second only holds when the scalar is non-negative (or the vector is $\mathbf{0}$).

4. Let $V$ be a 3-dimensional space. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ and $T = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be bases for $V$ with transition matrix from $S$ to $T$ equal to $Q = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & -4 \end{bmatrix}$.
Let $\mathbf{u} = 2\mathbf{v_1} + 3\mathbf{v_2} + \mathbf{v_3}$. Find real numbers $a, b, c$ such that $\mathbf{u} = a\mathbf{w_1} + b\mathbf{w_2} + c\mathbf{w_3}$. (10 pts)

Use the equation that $[\mathbf{u}]_T = Q[\mathbf{u}]_S$. The $S$ coordinate is $[\mathbf{u}]_S = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ so

$[\mathbf{u}]_T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$. This means that $\mathbf{u} = \mathbf{w_1} + 7\mathbf{w_2} + 3\mathbf{w_3}$ so $a = 1, b = 7, c = 3$.

5. Let $V$ be the 3-dimensional subspace of $\mathbb{R}^4$ with basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -3 \end{bmatrix} \right\}$.
Find an orthonormal basis for $V$. (14 pts)

Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -3 \end{bmatrix}$. Use the Gram-Schmidt formulas to find an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v_2} = \mathbf{u_2} - \left(\frac{\mathbf{v_1}\cdot\mathbf{u_2}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

$$\mathbf{v_3} = \mathbf{u_3} - \left(\frac{\mathbf{v_1}\cdot\mathbf{u_3}}{\mathbf{v_1}\cdot\mathbf{v_1}}\right)\mathbf{v_1} - \left(\frac{\mathbf{v_2}\cdot\mathbf{u_3}}{\mathbf{v_2}\cdot\mathbf{v_2}}\right)\mathbf{v_2} = \begin{bmatrix}2\\-1\\0\\-3\end{bmatrix} - \frac{2}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \frac{-4}{2}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}$$

This gives the orthogonal basis $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\right\}$. To get an orthonormal basis, divide each vector by its length to get $\left\{\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}0\\1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}\right\}$.

6. Let $L : P_3 \to M_{22}$ be the linear transformation with $\dim \operatorname{range} L = 4$.

   (a) Prove that $L$ is invertible. (6 pts)

   The range is a 4-dimensional subspace of $\mathbb{R}^4$, so it must be all of $\mathbb{R}^4$ so $L$ is onto.
   Using the formula $\dim\ker L + \dim\operatorname{range} L = \dim P_3$ and plugging in 4 for $\dim\operatorname{range} L$ and $\dim P_3$, we get that $\dim\ker L = 0$. Thus the kernel of $L$ is $\{0\}$ and $L$ is one-to-one. $L$ is both one-to-one and onto so it is invertible.

   (b) Suppose that $L(t^3) = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$ and $L(t) = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$. Find $L^{-1}\left(\begin{bmatrix}1\\2\\2\\1\end{bmatrix}\right)$. (8 pts)

$$L^{-1}\left(\begin{bmatrix}1\\2\\2\\1\end{bmatrix}\right) = L^{-1}\left(\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right) + 2L^{-1}\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right) = t^3 + 2t.$$

7. Let $L : M_{22} \to \mathbb{R}^2$ be a linear transformation $L\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = \begin{bmatrix}a+b\\c-d\end{bmatrix}$.

   (a) Find a basis for the kernel of $L$. (8 pts)

The kernel of $L$ is all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a+b=0$ and $c-d=0$. Thus it is all matrices of the form $\begin{bmatrix} a & -a \\ c & c \end{bmatrix} = a\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ so $\ker L = \operatorname{span}\left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right\}$. This set is linearly independent, so $\left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right\}$ is a basis for $\ker L$.

(b) Find a basis for the range of $L$. (6 pts)

The range of $L$ is all vectors of the form $\begin{bmatrix} a+b \\ c-d \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ so range $L = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\} = \mathbb{R}^2$. This has basis $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

(c) Is $L$ one-to-one? Is $L$ onto? (4 pts)

$L$ is not one-to-one as the kernel is nontrivial. It is onto as the range is all of $\mathbb{R}^2$.

(d) Find the representation of $L$ with respect to $S$ and $T$ where $S = \left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right\}$ and $T = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. (12 pts)

Plug each of the vectors from $S$ into $L$, then take coordinates with respect to $T$.

$L\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so the coordinate is $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

$L\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so the coordinate is $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

$L\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so the coordinate is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$L\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so the coordinate is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

These are the columns of the representation, so it is $\begin{bmatrix} 0 & -1 & 1 & 2 \\ 2 & 4 & 0 & -1 \end{bmatrix}$.