Exam 2 Solutions

1. Let V be the set of pairs of real numbers (x, y). Define the following operations on V:

$$(x, y) \oplus (x', y') = (x + x', xx' + yy')$$
$$r \odot (x, y) = (rx, y)$$

Check if V together with $\oplus$ and $\odot$ satisfy properties 3 and 5 from the definition of a vector space. The properties are repeated below.

(a) Property 3: There exists an element $\mathbf{0}$ in V such that $\mathbf{u} \oplus \mathbf{0} = \mathbf{0} \oplus \mathbf{u} = \mathbf{u}$ for any $\mathbf{u}$ in V.

If $\mathbf{u} = (x, y)$ and $\mathbf{0} = (e, f)$, $\mathbf{u} \oplus \mathbf{0} = (x, y) \oplus (e, f) = (x + e, xe + yf)$ but it also needs to be equal to $\mathbf{u} = (x, y)$, so we need $x + e = x$ and $xe + yf = y$, so $e = 0$ and $f = 1$. The pair of real numbers $(0, 1)$ is in the set V and has $(x, y) \oplus (0, 1) = (0, 1) \oplus (x, y) = (x, y)$ for any pair of real numbers (x, y). This property is therefore satisfied and the zero vector is $(0, 1)$.

(b) Property 5: $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$ for any $\mathbf{u}, \mathbf{v}$ in V and any real number c.

Let $\mathbf{u} = (x, y)$ and $\mathbf{v} = (x', y')$. The left side is $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot ((x, y) \oplus (x', y')) = c \odot (x + x', xx' + yy') = (c(x + x'), xx' + yy')$. The right side is $c \odot \mathbf{u} \oplus c \odot \mathbf{v} = c \odot (x, y) \oplus c \odot (x', y') = (cx, y) \oplus (cx', y') = (cx + cx', c^2xx' + yy')$. The second entries are not always equal so this property does not hold.

2. Let W be the set of all 2-vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that x and y are both greater than or equal to 0 or are both less than or equal to 0. Assume the regular addition and scalar multiplication in $\mathbb{R}^2$.

(a) Is W closed under addition? Why or why not?

W is not closed under addition. For example, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ are both in W, but their sum $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is not in W.

(b) Is W closed under scalar multiplication? Why or why not?

W is closed under multiplication. A 2-vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is in W if x, y have the same sign (or one or both of them is 0). Taking a scalar multiple will either keep the signs the same or change both signs, so the scalar multiples of any vector in W are also in W.

Another way to see that W is closed under multiplication is think of W as all vectors on the xy-plane with tail at the origin and head in either the first or third quadrant. The scalar multiples of these vectors are also in the first or third quadrant.

(c) Is W a subspace of $\mathbb{R}^2$?

No. It is not closed under addition so it is not a subspace.

3. Let W be the subspace of M_{22} which consists of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a - b = c - d$.

(a) Find a basis for W.

If we solve the equation $a - b = c - d$ for a we get that $a = b + c - d$ so W is all vectors of the form $\begin{bmatrix} b+c-d & b \\ c & d \end{bmatrix} = b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. The set $\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ spans W and is linearly independent so it is a basis for W. Note that if you solved for a different variable, the basis you get will look slightly different.

(b) What is the dimension of W?

The basis for W has size 3, so $\dim W = 3$.

(c) Find a basis for M_{22} which contains your basis from part (a).

M_{22} has dimension 4, so we need to add just one matrix to the basis from (a). To make sure the set stays linearly independent, the matrix added cannot be a linear combination of the first three matrices so it cannot be in W. Any matrix from M_{22} which is not in W will keep the set linearly independent and any linearly independent set of matrices in M_{22} is a basis for M_{22}. Therefore the fourth matrix can be any matrix which is not in W. For example, $\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ is a basis for M_{22} which contains the basis from (a).

4. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ be a set of vectors in a vector space V.
Let $W = \text{span } S$

Fill in the following blanks with $<, >, \leq, \geq, =$. Choose the best possible answer that can be determined from the given information.

(a) $\dim W \leq 4$

(b) If S is linearly independent, then $\dim W = 4$.

(c) If $\mathbf{v_2} = 3\mathbf{v_4} - \mathbf{v_1}$, then $\dim W < 4$.

(d) If S is a basis for V, then $\dim W = \dim V$.

(e) If S is linearly independent and does not span V, then $\dim V > 4$

(f) If S is linearly dependent and spans V, then $\dim V < 4$

(g) $\dim \operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_2} - 3\mathbf{v_4}\} = \dim W$.

5. Let $S = \{t^2 + t, t^2 - t, t^2, 1\}$ be a set of vectors in P_2.

   (a) Is S linearly independent? Why or why not?

   No. P_2 has dimension 3 so any set of more than three vectors is linearly dependent. There are 4 vectors in S so S must be linearly dependent.

   Another way to show S is linearly dependent is to write one of the polynomials in S as a linear combination of the other polynomials in S. For example $t^2 = \frac{1}{2}(t^2 + t) + \frac{1}{2}(t^2 - t)$, so the third one is a linear combination of the first two. Yet another way to show S is linearly

   dependent is to take a linear combination of the vectors in S and set them equal to 0 and show that there is a nontrivial solution. If $x(t^2+t)+y(t^2-t)+z(t^2)+w(1) = 0$ then $(x+y+z)t^2+(x-y)t+w = 0$ so we get the homogeneous linear system $x + y + z = 0, x - y = 0, w = 0$. This has coefficient matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The row operations $r_2 - r_1 \to r_2, -\frac{1}{2}r_2 \to r_2$ take the matrix to REF of $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
   There is no leading one in column 3, so there are infinite solutions to this system and hence the vectors are linearly dependent.

   (b) Does S span P_2? Why or why not?

   Yes. One way to do this is to show that $\dim \operatorname{span} S = 3$. Then $\dim P_2 = 3$ as well and span S is subspace of P_2, so span $S = P_2$ (see HW 7 Problem 2). If we take a linear combination of the vectors in S and set it equal to 0, we get the homogeneous linear system from part (a). The REF has 3 leading ones, so the span of S has dimension 3.

   Another way to show that S spans P_2 is show that for any $at^2 + bt + c$ in P_2, the linear system associated with $x(t^2 + t) + y(t^2 - t) + z(t^2) + w(1) = at^2 + bt + c$ has at least one solution. This is the system $x + y + z = a, x - y = b, w = c$. The coefficient matrix is the same as the matrix from part (a) which doesn't have a row of zeros in REF, so not matter what we pick for a, b, c there will be solutions.

(c) Find a basis for span S which is contained in S.

We will use the method from p.235 in the textbook. As in part (a), the equation $x(t^2+t)+y(t^2-t)+z(t^2)+w(1)=0$ gives us the system with coefficient matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This is row equivalent to $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which is in REF. There are leading ones in columns 1,2,and 4 so the first, second, and fourth vectors in S are a basis for the span of S, so the basis is $\{t^2+t, t^2-t, 1\}$.

Note that $\{t^2+t, t^2, 1\}$ and $\{t^2-t, t^2, 1\}$ are also correct answers to this problem. We just need to delete one vector and it has to be one of the ones which can be written as a linear combination of the others, so we can delete the first, second, or third vector.

6. Let $A = \begin{bmatrix} 2 & 2 & 1 & -5 & -5 \\ 3 & 3 & 2 & -1 & 5 \\ -4 & -4 & 0 & 7 & 2 \\ -2 & -2 & 0 & 7 & 8 \end{bmatrix}$. The RREF of A is $\begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Find the rank and nullity of A.

The rank is 3 and the nullity is 2 (there are three columns with leading ones in the RREF of A and 2 columns without leading ones).

(b) Is $\begin{bmatrix} 5 & 5 & 5 & -5 & 5 \end{bmatrix}$ in the row space of A? Why or why not?

This vector is not in the row space of A. The nonzero rows of the RREF of A are a basis for the row space of A, so we need to see if we can write $\begin{bmatrix} 5 & 5 & 5 & -5 & 5 \end{bmatrix} =$ $x\begin{bmatrix} 1 & 1 & 0 & 0 & 3 \end{bmatrix} + y\begin{bmatrix} 0 & 0 & 1 & 0 & -1 \end{bmatrix} + z\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ for some x, y, z. This gives us the linear system $5 = x, 5 = y, -5 = z, 3x - y + 2z = 5$. This system has no solutions as $3(5) - (5) + 2(-5) = 0 \neq 5$. You can also check this using the original rows of A, but the rows in the RREF are much easier to work with.

(c) Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ a basis for the column space of A? Why or why not?

This is not a basis for the column space of A. Any vector in the span of these three vectors has a 0 for the fourth entry so column 1 of A is not in the span of these vectors. The vectors clearly don't span the column space of A, so they are not a basis. These vectors may not even be in the column space of A.

(d) Is the vector $\begin{bmatrix} 0 \\ -3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ in the null space of A?

Yes. Multiply either A or the RREF of A by this vector and see if the result is the zero vector. The multiplication is slightly easier using the RREF of A.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-3+0+0+3 \\ 0+0+1+0+-1 \\ 0+0+0+-2+2 \\ 0+0+0+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ or}$$

$$\begin{bmatrix} 2 & 2 & 1 & -5 & -5 \\ 3 & 3 & 2 & -1 & 5 \\ -4 & -4 & 0 & 7 & 2 \\ -2 & -2 & 0 & 7 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-6+1+10-5 \\ 0-9+2+2+5 \\ 0+12+0-14+2 \\ 0+6+0-14+8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

You can also do this by finding a basis for the null space and seeing if the vector is a linear combination of the basis vectors.