Math 3333
Spring 2015
Midterm 2

Name:____

Problem	**Points**
Problem 1 (16pts)	
Problem 2 (15pts)	
Problem 3 (15pts)	
Problem 4 (21pts)	
Problem 5 (15pts)	
Problem 6 (18pts)	
Total	

All vector spaces on this exam are real vector spaces. The definition of a real vector space, as it appears in your textbook, is provided below.

Definition: A **real vector space** is a set V of elements on which we have two operations $\oplus$ and $\odot$ defined with the following properties.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are any elements in V, then $\mathbf{u} \oplus \mathbf{v}$ is in V. (We say that V is **closed** under the operation $\oplus$).

(1) $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ for all $\mathbf{u}, \mathbf{v}$ in V.

(2) $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V.

(3) There exists an element $\mathbf{0}$ in V such that $\mathbf{u} \oplus \mathbf{0} = \mathbf{0} \oplus \mathbf{u} = \mathbf{u}$ for any $\mathbf{u}$ in V.

(4) For each $\mathbf{u}$ in V there exists an element $-\mathbf{u}$ in V such that $\mathbf{u} \oplus -\mathbf{u} = -\mathbf{u} \oplus \mathbf{u} = \mathbf{0}$.

(b) If $\mathbf{u}$ is any element in V and c is any real number, then $c \odot \mathbf{u}$ is in V (i.e. V is closed under the operation $\odot$).

(5) $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$ for any $\mathbf{u}, \mathbf{v}$ in V and any real number c.

(6) $(c + d) \odot \mathbf{u} = c \odot \mathbf{u} \oplus d \odot \mathbf{u}$ for any $\mathbf{u}$ in V and any real numbers c and d.

(7) $c \odot (d \odot \mathbf{u}) = (cd) \odot \mathbf{u}$ for any $\mathbf{u}$ in V and any real numbers c and d.

(8) $1 \odot \mathbf{u} = \mathbf{u}$ for any $\mathbf{u}$ in V.

1. Let V be the set of pairs of real numbers (x, y). Define the following operations on V:

$$(x, y) \oplus (x', y') = (x + x', xx' + yy')$$

$$r \odot (x, y) = (rx, y)$$

Check if V together with $\oplus$ and $\odot$ satisfy properties 3 and 5 from the definition of a vector space. The properties are repeated below.

(a) Property 3: There exists an element $\mathbf{0}$ in V such that $\mathbf{u} \oplus \mathbf{0} = \mathbf{0} \oplus \mathbf{u} = \mathbf{u}$ for any $\mathbf{u}$ in V. (8 pts)

(b) Property 5: $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot \mathbf{u} \oplus c \odot \mathbf{v}$ for any $\mathbf{u}, \mathbf{v}$ in V and any real number c. (8 pts)

2. Let W be the set of all 2-vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that x and y are both greater than or equal to 0 or are both less than or equal to 0. Assume the regular addition and scalar multiplication in $\mathbb{R}^2$.

   (a) Is W closed under addition? Why or why not? (6 pts)

   (b) Is W closed under scalar multiplication? Why or why not? (6 pts)

   (c) Is W a subspace of $\mathbb{R}^2$? (3 pts)

3. Let W be the subspace of M_{22} which consists of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a - b = c - d$.

(a) Find a basis for W. (7 pts)

(b) What is the dimension of W? (3 pts)

(c) Find a basis for M_{22} which contains your basis from part (a). (5 pts)

4. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ be a set of vectors in a vector space V.
Let $W = \text{span } S$

Fill in the following blanks with $<, >, \leq, \geq, =$.

Choose the best possible answer that can be determined from the given information. You do not need to show work or explain your answer for this problem. (3 pts each part)

(a) $\dim W$ ______ 4.

(b) If S is linearly independent, then $\dim W$ ______ 4.

(c) If $\mathbf{v_2} = 3\mathbf{v_4} - \mathbf{v_1}$, then $\dim W$ ______ 4.

(d) If S is a basis for V, then $\dim W$ ______ $\dim V$.

(e) If S is linearly independent and does not span V, then $\dim V$ ______ 4.

(f) If S is linearly dependent and spans V, then $\dim V$ ______ 4.

(g) $\dim \text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_2} - 3\mathbf{v_4}\}$ ______ $\dim W$.

5. Let $S = \{t^2 + t, t^2 - t, t^2, 1\}$ be a set of vectors in P_2.

(a) Is S linearly independent? Why or why not? (5 pts)

(b) Does S span P_2? Why or why not? (5 pts)

(c) Find a basis for span S which is contained in S. (5 pts)

6. Let $A = \begin{bmatrix} 2 & 2 & 1 & -5 & -5 \\ 3 & 3 & 2 & -1 & 5 \\ -4 & -4 & 0 & 7 & 2 \\ -2 & -2 & 0 & 7 & 8 \end{bmatrix}$. The RREF of A is $\begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Find the rank and nullity of A. (4 pts)

(b) Is $\begin{bmatrix} 5 & 5 & 5 & -5 & 5 \end{bmatrix}$ in the row space of A? Why or why not? (5 pts)

(c) Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ a basis for the column space of A? Why or why not? (5 pts)

(d) Is the vector $\begin{bmatrix} 0 \\ -3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ in the null space of A? Why or why not? (4 pts)