Linear Algebra - Exam 1 Solutions

1. Is the vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$?

   Why or why not? (8 pts)

   No. We need to check if there are constants x, y, z such that
   $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$. This gives us the system of equations
   $x + 2y - z = 1, x + 2y - z = 2, -x + y + 2z = -1$. This system has no
   solutions as $x + 2y - z$ cannot equal both 1 and 2. The first vector is therefore
   not a linear combination of the other three.

2. Suppose A and B are $n \times n$ matrices and that A is symmetric and B is skew
   symmetric. Determine if AB is symmetric, skew symmetric, both, or neither.

   Either give a proof (if your answer is symmetric, skew symmetric, or both), or
   find an example which shows that AB can be neither. (12 pts)

   Neither. As A is symmetric and B is skew symmetric, $A^T = A$ and $B^T = -B$.
   Then $(AB)^T = B^T A^T = -BA$. This is not necessarily equal to AB or $-(AB)$
   so it is not obviously symmetric or skew symmetric. We therefore look for an
   example to show can be neither. One possible example is $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and

   $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$. This example shows
   that for A symmetric and B skew symmetric, the product AB does not have
   to be symmetric or skew symmetric.

3. Let $A = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 5 & 0 & -1 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 1 & 0 \end{bmatrix}$. Compute the following determinants. (24 pts)

   (a) $\det(A)$

   There are many different possible methods to compute this, we will just
   use one here. Using cofactor expansion along column 1,
   $\det(A) = 2 \det\left(\begin{bmatrix} 1 & 2 & 0 \\ 5 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \right)$. Cofactor expansion along column 3
   gives $\det(A) = 2(-)(-1) \det\left(\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \right) = 2(1 - (-2)) = 6$.

1

(b) $\det(2A^{-1})$

Using properties of determinants and the answer from part (a),
$\det(2A^{-1}) = 2^4 \det(A^{-1}) = 2^4/\det(A) = 2^4/6 = 8/3$.

(c) $\det(A^T A)$

Again using properties of determinants and the answer from part (a),
$\det(A^T A) = \det(A^T)\det(A) = \det(A)\det(A) = 36$.

(d) $\det(A^T - A)$

This cannot be done using properties of determinants, so we need to
compute $A^T - A$.
$$A^T - A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 5 & 0 & -1 \\ 2 & 0 & 3 & 1 \\ 0 & -1 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 5 & 0 & -1 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$ The
row operations $r_1 \leftrightarrow r_2, r_3 \leftrightarrow r_4$ will take this to the diagonal matrix
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$. The diagonal matrix has determinant 9. The row
operations we did were both type 1 which changes the sign, but since we
did 2 of them the sign changed twice so $A^T - A$ also has determinant 9.

4. Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 1 & 1 & 4 \end{bmatrix}$.

(a) Find A^{-1} or show that A is not invertible. (12 pts)

Start with the matrix $[A : I]$ and do row operations to get $[I : A^{-1}]$ or
until it becomes clear that the RREF of A is not I. Starting with
$\begin{bmatrix} 1 & 0 & 3 & | & 1 & 0 & 0 \\ 2 & 1 & 6 & | & 0 & 1 & 0 \\ 1 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix}$, we do the row operations
$r_2 - 2r_1 \to r_2, r_3 - r_1 \to r_3, r_3 - r_2 \to r_3, r_1 - 3r_3 \to r_3$. The resulting
matrix is $\begin{bmatrix} 1 & 0 & 0 & | & -2 & 3 & -3 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & -1 & 1 \end{bmatrix}$ so $A^{-1} = \begin{bmatrix} -2 & 3 & -3 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.

(b) Use your answer to part (a) to find the solutions to $A^2\mathbf{x} = \mathbf{b}$ where
$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. (12 pts)

Since A is invertible, we can solve $A^2\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ by multiplying both
sides of the equation by $(A^2)^{-1} = (A^{-1})^2$ on the left. The only solution to

2

this linear system is

$$\mathbf{x} = (A^{-1})^2\mathbf{b} = A^{-1}(A^{-1}\mathbf{b}) = \begin{bmatrix} -2 & 3 & -3 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\left(\begin{bmatrix} -2 & 3 & -3 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) =$$

$$\begin{bmatrix} -2 & 3 & -3 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

5. Suppose A is a 4×3 matrix and that the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution for some 4-vector $\mathbf{b}$.

   (a) If possible, find the RREF of A. Otherwise describe what can be said about the RREF of A from the given information. (6 pts)

   As $A\mathbf{x} = \mathbf{b}$, the RREF of A must have a leading one in each column. There are three columns so there are also three leading ones. There is therefore a zero row and it must be at the bottom, so row 4 is all 0's. The leading ones move to the right, so they must be in the 11,22, and 33 positions. The columns with leading ones have all other entries equal to 0. The RREF is therefore $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

   Note: A is not square, so it does not make sense to talk about the inverse of A or the determinant of A, as these things are only defined for square matrices.

   (b) If $\mathbf{c}$ is another 4-vector, what are the possible numbers of solutions to the linear system $A\mathbf{x} = \mathbf{c}$? (6 pts)

   The possible numbers of solutions are 0 or 1. The linear system has augmented matrix $[A : \mathbf{c}]$. If you do the row operations which take A to RREF, the number of solutions will depend on what happens to $\mathbf{c}$. If the result has a 0 in the 4th entry, there will be 1 solution. If it has a nonzero number in the 4th entry there will be 0 solutions. It cannot have infinite solutions since the columns in RREF of A all have leading ones.

6. Consider the linear system:

$$-a - 2b + d - 2e = 2$$
$$a + 2b + 3c - 5d + 15e = -11$$
$$2a + 4b + c - d + 6e = 7$$

(a) Find the augmented matrix of the linear system. (6 pts)

$$\left[\begin{array}{ccccc|c} -1 & -2 & 0 & 1 & -2 & 2 \\ 1 & 2 & 3 & -5 & 15 & -11 \\ 2 & 4 & 1 & -1 & 6 & 7 \end{array}\right]$$

(b) One of the following is the reduced row echelon form (RREF) of the augmented matrix. Circle it.

(8 pts)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 4 \\ 0 & 1 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -1 & 6 \end{array}\right] \qquad \left[\begin{array}{ccccc|c} 1 & -3 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -1 & 6 \end{array}\right] \qquad \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & -11 \\ 0 & 0 & 0 & 1 & -1 & 7 \end{array}\right]$$

One way to do this is to do row operations until it becomes clear which matrix it is. From REF, you can see where the leading ones are and what the last row is, which is enough to tell that the RREF is the matrix on the bottom left.

Another option is to guess, then check it by seeing if the solutions it gives you are solutions to the original equations. The top left matrix is in RREF, so this is not going to be the answer. In each row of the augmented matrix, the entry in column 2 is twice the entry in column 1. This tells us that row 1 of RREF will start with 1,2 and the other entries in columns 1 and 2 are 0's. This narrows things down to the bottom two matrices. The last column in the matrix on the right matches the last column of the augmented matrix. It's unlikely that the last column wouldn't have been changed by the row operations, so you might guess that it's the matrix on the bottom left. You can confirm this by finding the solutions to both linear systems. The ones from the matrix on the right don't work in the original equations, but the ones from the matrix on the left do.

(c) Find all solutions to the linear system. Write your answer as a vector.

(8 pts)

Use the RREF from part (b). There are no leading ones in the columns corresponding to b and e so these variables can be anything. Then $a = 4 - 2b - e, c = 5 - 3e, d = 6 + e$ so the solutions are all vectors of the

form $\begin{bmatrix} 4 - 2b - e \\ b \\ 5 - 3e \\ 6 + e \\ e \end{bmatrix}$ where b, e are anything.

4