Review for Final Exam

1. If A is an invertible matrix $n \times n$ matrix, which of the following must be true?

   (a) $\det(A) = 1$.

   (b) The columns of A are an orthogonal set in $\mathbb{R}^n$.

   (c) 0 is not an eigenvalue of A.

   (d) The reduced row echelon form of A is I_n.

   (e) A is diagonalizable.

   (f) The rows of A are a basis for $\mathbb{R}_n$.

   Answer: c,d,f

2. Let $A = \begin{bmatrix} 2 & 3 & 0 & 0 & -6 \\ 0 & 0 & 0 & 1 & 5 \\ -1 & 0 & 6 & 3 & 3 \\ 0 & 1 & 4 & 2 & 0 \end{bmatrix}$.

   (a) Find the RREF of A.

   The following row operations will put A into RREF. $r_1 \leftrightarrow r_3$, $-r_1 \to r_1$, $r_2 \leftrightarrow r_4$, $r_3 - 2r_1 \to r_3$, $r_3 - 3r_2 \to r_3$, $r_3 \leftrightarrow r_4$, $r_1 + 3r_3 \to r_1$, $r_2 - 2r_3 \to r_3$.
   The resulting matrix is $\begin{bmatrix} 1 & 0 & -6 & 0 & 12 \\ 0 & 1 & 4 & 0 & -10 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (b) What are the rank and nullity of A?

   The rank is 3 and nullity is 2. The rank is the number of leading ones in RREF and the nullity is the number of columns without leading ones in RREF and the sum of the two equals the number of columns.

   (c) Find a basis for the null space of A.

   The nullity is 2 so a basis for the null space will contain two vectors. The null space is the same as the solution space of $A\mathbf{x} = \mathbf{0}$ which can be found using the RREF of A. If we label the variables as a, b, c, d, e then we see from the RREF that the system $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $a - 6c + 12e = 0, b + 4c - 10e = 0, d + 5e = 0$. Columns 3 and 5 do not contain leading ones so the variables c and e can be anything and the other variables can be written in terms of c and e. In particular $a = 6c - 12e, b = -4c + 10e, d = -5e$. The null space is all vectors of the form $\begin{bmatrix} 6c-12e \\ -4c+10e \\ c \\ -5e \\ e \end{bmatrix} = c\begin{bmatrix} 6 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e\begin{bmatrix} -12 \\ 10 \\ 0 \\ -5 \\ 1 \end{bmatrix}$ and has basis $\left\{\begin{bmatrix} 6 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 10 \\ 0 \\ -5 \\ 1 \end{bmatrix}\right\}$.

(d) Find a basis for the column space of A.

The rank is 3 so both the column space and row space of A have dimension 3 so the bases in this part and the next part will have size 3. The leading ones in RREF are in columns $1, 2, 4$ so columns $1, 2, 4$ of the original matrix will be a basis for the column space so a basis is $\left\{\begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}\right\}$.

(e) Find a basis for the row space of A.

The nonzero rows from RREF form a basis for the row space so a basis is $\{\begin{bmatrix} 1 & 0 & -6 & 0 & 12 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 4 & 0 & -10 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 5 \end{bmatrix}\}$.

(f) Let $\mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 11 \\ 7 \end{bmatrix}$. Prove that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Find all the solutions to $A\mathbf{x} = \mathbf{b}$.

$\begin{bmatrix} 2 & 3 & 0 & 0 & -6 \\ 0 & 0 & 0 & 1 & 5 \\ -1 & 0 & 6 & 3 & 3 \\ 0 & 1 & 4 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 11 \\ 7 \end{bmatrix}$. The other solutions to $A\mathbf{x} = \mathbf{b}$ will look like the particular solution $\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$ plus solutions to the homogeneous linear system. Therefore the solutions are all vectors of the form $\begin{bmatrix}1+6c-12e\\1-4c+10e\\1+c\\1-5e\\1+e\end{bmatrix}$.

3. Let A and B be $n \times n$ matrices such that $\det(A) = 4, \det(B) = -1$.

   (a) What is $\det(A^2B^T)$?

   $\det(A^2B^T) = \det(A)\det(A)\det(B^T) = \det(A)^2\det(B) = -16.$

   (b) What are the rank and nullity of A^2B^T?

   A^2B^T is invertible as its determinant is nonzero so it has rank n and nullity 0.

   (c) Let $\mathbf{c}$ be a fixed vector in $\mathbb{R}^n$. How many solutions does $A^2B^T\mathbf{x} = \mathbf{c}$ have? What are the solutions?

   As A^2B^T is invertible, any linear system $A^2B^T\mathbf{x} = \mathbf{c}$ has exactly one solution which looks like $\mathbf{x} = (A^2B^T)^{-1}\mathbf{c}$. This can also be written as $(B^T)^{-1}(A^2)^{-1}\mathbf{c}$ or as $(B^{-1})^T(A^{-1})^2\mathbf{c}$.

4. For what values of a is $\begin{bmatrix}a^2\\-3a\\-2\end{bmatrix}$ in span $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\4\end{bmatrix}\right\}$?

If $\begin{bmatrix}a^2\\-3a\\-2\end{bmatrix}$ is in the span of these vectors, then we can write $\begin{bmatrix}a^2\\-3a\\-2\end{bmatrix} = x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}0\\1\\1\end{bmatrix} + z\begin{bmatrix}1\\3\\4\end{bmatrix}$ for some x, y, z. Treating a as a constant, this is the linear system $a^2 = x + z, -3a = 2x + y + 3z, -2 = 3x + y + 4z$. This has augmented matrix $\left[\begin{array}{ccc|c}1&0&1&a^2\\2&1&3&-3a\\3&1&4&-2\end{array}\right]$. The row operations $r_2 - 2r_1 \to r_2, r_3 - 3r_1 \to r_3, r_3 - r_2 \to$

r_3 will put the left side of the matrix into RREF. The resulting matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 1 & a^2 \\ 0 & 1 & 1 & -3a-2a^2 \\ 0 & 0 & 0 & -2-a^2+3a \end{array}\right]$. This has solutions if and only if $0 = -2 - a^2 + 3a$. Equivalently, $a^2 - 3a + 2 = 0$ and this factors as $(a-2)(a-1) = 0$ so $a = 1, 2$.

Note that the span of the three vectors was not all of $\mathbb{R}^3$. This is because the last vector is the sum of the first two. So we could have deleted the third vector without changing the span and done the problem using just the first two vectors.

We can check this answer by making sure that when $a = 1$ or $a = 2$ the vector really is in the span. When $a = 1$, $\begin{bmatrix} a^2 \\ -3a \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. When $a = 2$, $\begin{bmatrix} a^2 \\ -3a \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 14 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

5. Determine if the following statements are true or false. Give a proof or counterexample.

   (a) If U and W are subspaces of a vector space V and $\dim U < \dim W$, then U is a subspace of W.

   False. For U to be a subspace of W it would have to be contained in W. This does not always have to be the case. For example, take $V = \mathbb{R}^3$ and U to be a 1-dimensional subspace and W to be a 2-dimensional subspace. The 1-dimensional subspaces of $\mathbb{R}^3$ are exactly the lines through the origin and the 2-dimensional subspaces are planes though the origin. Given a line and plane though the origin, it is not true that the line must be on the plane. For example, take U to be the z-axis which is all vectors of the form $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$ and take W to be the xy-plane which is all vectors of the form $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$.

   (b) Any subspace of $\mathbb{R}^3$ which contains the vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ must also contain the vector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

True. Subspaces are closed under scalar multiplication and addition so if a subspace contains a set of vectors then it must contain all linear combinations of those vectors. In this case, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is a linear combination of the other two vectors.

6. Let S be the following set of vectors in $\mathbb{R}^4$.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(a) Find a subset of S which is a basis for span S.

One way to do this is to construct a matrix whose columns are the vectors in S. Then put the matrix in RREF and the columns with leading ones will correspond to the vectors of S which are the basis for span S. The matrix would be $\begin{bmatrix} 1 & 0 & 3 & -7 & 1 \\ 0 & 2 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 6 & 6 & 11 & 0 \end{bmatrix}$. We do not actually need to go all the way to RREF, just far enough that it is clear which columns will have leading ones. Doing the row operations $r_4 - r_1 \to r_4, r_4 - 3r_2 \to r_4, r_4 - r_3 \to r_4$ we get $\begin{bmatrix} 1 & 0 & 3 & -7 & 1 \\ 0 & 2 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The leading ones will be in columns $1, 2, 5$ so we take the first, second, and fifth vectors of S and our basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$.

(b) Does S contain a basis for $\mathbb{R}^4$? Is S contained in a basis for $\mathbb{R}^4$?

S does not contain a basis for $\mathbb{R}^4$ because if it did then the span of S would be all of $\mathbb{R}^4$ but the span of S is only 3-dimensional.

S is not contained in a basis for $\mathbb{R}^4$ because it is too big. Any basis for $\mathbb{R}^4$ will contain exactly 4 vectors so it will not be able to contain a set of 5 vectors.

7. Fix a real number λ and a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$. Determine if the following sets are subspaces of M_{nn}.

   (a) The set of all $n \times n$ matrices with eigenvalue λ.

   This is not a subspace of M_{nn}. Suppose we try to check closed under addition. If A and B are matrices with eigenvalue λ, then $A\mathbf{v_1} = \lambda\mathbf{v_1}$ for some nonzero vector $\mathbf{v_1}$ and $B\mathbf{v_2} = \lambda\mathbf{v_2}$ for another nonzero vector $\mathbf{v_2}$. Since $\mathbf{v_1}$ and $\mathbf{v_2}$ may not be the same vector, there is no obvious way to prove that λ would have to be an eigenvalue of $A + B$. We therefore look for a counterexample to show that it doesn't have to be.

   Suppose $\lambda = 0$. Then the matrices which have $\lambda = 0$ as an eigenvalue are exactly the matrices for which $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. This is the set of matrices which are not invertible. This is not closed under addition as $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both in the set of non-invertible matrices but their sum is the identity which is invertible. These two matrices both have eigenvalue 0 but their sum does not. This example shows that this set does not have to be a subspace because it is not necessarily closed under addition.

   Note: When $\lambda \neq 0$, it's still not a subspace, since it does not contain the zero matrix.

   (b) The set of all $n \times n$ matrices with eigenvector $\mathbf{v}$.

   This is a subspace. As $\mathbf{v} \neq \mathbf{0}$, it is an eigenvector of the zero matrix O since $O\mathbf{v} = \mathbf{0} = 0\mathbf{v}$. This set therefore contains the zero matrix so it is nonempty. Next check closed under addition. Suppose A and B both have eigenvector $\mathbf{v}$. Then $A\mathbf{v} = \lambda_1\mathbf{v}$ and $B\mathbf{v} = \lambda_2\mathbf{v}$ for some scalars λ_1, λ_2. Their sum has $(A+B)\mathbf{v} = A\mathbf{v}+B\mathbf{v} = \lambda_1\mathbf{v}+\lambda_2\mathbf{v} = (\lambda_1+\lambda_2)\mathbf{v}$. Therefore $A+B$ also has eigenvector $\mathbf{v}$, with associated eigenvalue $\lambda_1+\lambda_2$. Next check closed under scalar multiplication. If A has eigenvector $\mathbf{v}$ then $A\mathbf{v} = \lambda_1\mathbf{v}$. For any real number r, $(rA)\mathbf{v} = r(A\mathbf{v}) = r(\lambda_1\mathbf{v}) = (r\lambda_1)\mathbf{v}$ so rA has $\mathbf{v}$ as an eigenvector with associated eigenvalue $r\lambda_1$.

8. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$. Determine if $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v}$ is an inner product on $\mathbb{R}^2$. Either show that it satisfies all four properties of an inner product or give an example of vectors that show it fails one of the properties.

   This is not an inner product. It satisfies properties 1, 3, and 4 of inner products but fails property 2 that $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$ for all vectors $\mathbf{u}, \mathbf{v}$. For example, consider the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

$$\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1$$

9. Let $S = \{t^2, t, 1\}$ be the standard basis for P_2. Define an inner product on P_2 by $(p(t), q(t)) = [p(t)]_S \cdot [q(t)]_S$. Let W be the subspace of all polynomials $p(t)$ in P_2 such that $p(2) = 0$.

   (a) Find an orthogonal basis for W.

   Start by finding any basis for W. Is $p(t)$ is in P_2 then it looks like $p(t) = at^2 + bt + c$. Then $p(2) = 4a + 2b + c$ so W is the set of all polynomials of the form $at^2 + bt + c$ such that $4a + 2b + c = 0$. If we solve for c we get $c = -4a - 2b$ so W is the set of all polynomials of the form $at^2 + bt - 4a - 2b = a(t^2 - 4) + b(t - 2)$. This is spanned by $\{t^2 - 4, t - 2\}$ and these are linearly independent vectors so they are a basis for W.

   Next transform the basis to an orthogonal basis using Gram-Schmidt. This maybe slightly easier if we reorder the basis to be $\{t - 2, t^2 - 4\}$. Let $\mathbf{u_1} = t - 2, \mathbf{u_2} = t^2 - 4$. The new basis will be $\{\mathbf{v_1}, \mathbf{v_2}\}$. Then $\mathbf{v_1} = \mathbf{u_1} = t - 2$ and $\mathbf{v_2} = -\frac{(\mathbf{u_2}, \mathbf{v_1})}{(\mathbf{v_1}, \mathbf{v_1})}\mathbf{v_1} + \mathbf{u_2}$. The inner products are $(\mathbf{u_2}, \mathbf{v_1}) = (t^2 - 4, t - 2) = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = 8$, $(\mathbf{v_1}, \mathbf{v_1}) = (t - 2, t - 2) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = 5$. Therefore $\mathbf{v_2} = -\frac{8}{5}(t - 2) + (t^2 - 4)$. If we want to avoid fractions, we can replace this with a scalar multiple, in particular 5 times this vector. So we take $\mathbf{v_2} = -8(t - 2) + 5(t^2 - 4) = 5t^2 - 8t - 4$. The orthogonal basis is $\{t - 2, 5t^2 - 8t - 4\}$.

Note: There are a lot of other possible correct answers to this problem. For example, if we solved for a or b in the beginning instead of c, then we would get a different basis. Or if we had written the basis in the other order or not cleared out fractions, the result of the G-S process would be different. Any set of two nonzero polynomials which are in W and have inner product 0 will be an orthogonal basis for W. So if you got a slightly different answer, say $\{p(t), q(t)\}$ with $p, q \neq 0$, you can check it by seeing if $p(2) = 0, q(2) = 0$ and $(p(t), q(t)) = 0$.

(b) Find a basis for $W^\perp$.

$W^\perp$ is the set of all vectors in P_2 which are orthogonal to every vector in W. If $at^2 + bt + c$ is orthogonal to a basis for W, then it will be orthogonal to all vectors in W. So we only need to find vector in P_2 which are orthogonal to a basis for W. We can use any basis for W to do the problem. Here we will take the basis $\{t^2 - 4, t - 2\}$ from the beginning of the solution to part (a). For $at^2 + bt + c$ to be orthogonal to $t^2 - 4$, we need $0 = (at^2 + bt + c, t^2 - 4) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = a - 4c$. To be orthogonal to $t - 2$, we need $0 = (at^2 + bt + c, t - 2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = b - 2c$. The vectors in $W^\perp$ are of the form $at^2 + bt + c$ with $a - 4c = 0$ and $b - 2c = 0$. This is a homogeneous linear system and the solutions are that c can be anything and $b = 2c, a = 4c$. So $W^\perp$ is all vectors of the form $4ct^2 + 2ct + c = c(4t^2 + 2t + 1)$. A basis for this space is $\{4t^2 + 2t + 1\}$.

As a check, we verify that $\dim W + \dim W^\perp = \dim P_2$. From part (a), $\dim W = 2$ and from the above $\dim W^\perp = 1$ and $\dim P_2 = 3$ so the equation is satisfied.

10. Let A be a fixed $n \times n$ matrix. Define $L : M_{nn} \to M_{nn}$ to be $L(X) = AX - XA$.

(a) Prove that L is a linear transformation.

$L(X + Y) = A(X + Y) - (X + Y)A = AX + AY - XA - YA = (AX - XA) + (AY - YA) = L(X) + L(Y)$ and $L(rX) = ArX - rXA = r(AX - XA) = rL(X)$ so L is a linear transformation.

(b) Is L one-to-one? Is L onto?

L is not one-to-one. To check if it is one-to-one, we need to check if $\ker L = \{\mathbf{0}\}$. The kernel is all matrices X in M_{nn} such that $AX - XA = \mathbf{0}$, or all X with $AX = XA$. Even though we do not know what A is, there is at least one nonzero matrix which is guaranteed to have this property. If we take $X = I$, the $n \times n$ identity, then $AX = AI = A$ and $XA = IA = A$ so I is in the kernel of L. As the kernel contains more than just the zero matrix, the map is not one-to-one.

L is also not onto. This is because $\dim M_{nn} = \dim \ker L + \dim \text{range } L$. As the kernel has dimension greater than 0, the dimension of the range must be less than the dimension of M_{nn} (which is n^2). Therefore the range cannot be all of M_{nn} so L is not onto.

11. Let $L : P_3 \to \mathbb{R}^4$ be the linear transformation $L(p(t)) = \begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \\ p'''(0) \end{bmatrix}$. Prove that L is invertible and find L^{-1}.

If $p(t)$ is in P_3, it looks like $p(t) = at^3 + bt^2 + ct + d$. Then $p(0) = d$, $p'(t) = 3at^2 + 2bt + c$ so $p'(0) = c$, $p''(t) = 6at + 2b$ so $p''(0) = 2b$, and $p'''(t) = 6a$ so $p'''(0) = 6a$. We can therefore rewrite L as $L(at^3 + bt^2 + ct + d) = \begin{bmatrix} d \\ c \\ 2b \\ 6a \end{bmatrix}$. We can show L is invertible and find its inverse using a representation. To do this, we need to pick bases for P_3 and $\mathbb{R}^4$. Take $S = \{t^3, t^2, t, 1\}$ and T to be the standard basis for $\mathbb{R}^4$. Then $L(t^3) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}$, $L(t^2) = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, $L(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $L(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Taking coordinated with respect to T does not do anything as T is the standard basis for $\mathbb{R}^4$. The representation of L with respect to S and T is

$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}$. Then $\det(A) = 12$ which is nonzero so A is invertible and thus L is also invertible. The inverse of A is $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/6 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. $L^{-1} : \mathbb{R}^4 \to P_3$ has the property that $[L^{-1}(\mathbf{v})]_S = A^{-1}[\mathbf{v}]_T$. Note that as T is the standard basis for $\mathbb{R}^4$, $[\mathbf{v}]_T = \mathbf{v}$ so $[L^{-1}(\mathbf{v})]_S = A^{-1}\mathbf{v}$. If we write $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ then $\left[L^{-1}\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) \right]_S = \begin{bmatrix} 0 & 0 & 0 & 1/6 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} (1/6)d \\ (1/2)c \\ b \\ a \end{bmatrix}$. Therefore $L^{-1}\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = \frac{1}{6}dt^3 + \frac{1}{2}ct^2 + bt + a$.

Note: This problem is actually a little easier if you pick $S = \{1, t, t^2, t^3\}$, as the representation is diagonal so the determinant and inverse are easier to compute. Any choice of bases will give you the same result in the end though.

12. Let $L : P_3 \to M_{22}$ be the linear transformation $L(at^3 + bt^2 + ct + d) = \begin{bmatrix} a-c & 2c+d \\ b+d & 2a-b \end{bmatrix}$.

   (a) Find bases for the kernel and range of L.

   To be in the kernel of L, a polynomial $at^3 + bt^2 + ct + d$ must have $a - c = 0, 2c + d = 0, b + d = 0, 2a - b = 0$. This forces $a = c, d = -2c, b = -d = 2c$ and the kernel is all polynomials of the form $ct^3 + 2ct^2 + ct - 2c$ which has basis $\{t^3 + 2t^2 + t - 2\}$.

   Using that $\dim \ker L + \dim \operatorname{range} L = \dim P_3$ we get that the range will have dimension 3. The range is all matrices of the form $\begin{bmatrix} a-c & 2c+d \\ b+d & 2a-b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} + c\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. These four matrices span the range but are not a basis for the range as the range only has dimension 3. One of the matrices must be a linear combination of the others.

We see that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (1/2)\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} + (1/2)\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$ so we can delete the fourth matrix without changing the span. We thus get that $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\right\}$ is a basis for the range of L.

Note: In this case, any one of the four matrices that span the range can be written as linear combination of the other three, so we can delete any one of them to get a basis for the range.

(b) Find the representation of L with respect to the bases $S = \{t^3, t^2, t, 1\}$ and $T = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$L(t^3) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, L(t^2) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, L(t) = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}, L(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The coordinate vectors with respect to T are $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ respectively. Putting these together, we get that the representation is $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & -1 & 0 & 0 \end{bmatrix}$.

(c) Let $S' = \{t^3, t^3 - t^2, t^3 + t^2 - t, t^3 + t^2 + t - 1\}$ and let

$$T' = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Find the representation of L with respect to S' and T' two different ways: directly and using transition matrices.

Method 1: To compute the representation directly, we need to plug the vectors from S' into L. $L(t^3) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, L(t^3 - t^2) = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}, L(t^3 + t^2 - t) = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}, L(t^3 + t^2 + t - 1) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. We then need to find the coordinate vectors of each of these with respect to T'. We're looking for x, y, z, w such that $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + w\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. Equivalently, we're solving the system $x - w = 1, y + z =$

$0, -y = 0, x = 2$ which has solution $x = 2, y = 0, z = 0, w = 1$ so the coordinate vector is $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Similarly, for $\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$ we're solving $x - w = 1, y + z = 0, -y = -1, x = 3$ which has solution $x = 3, y = 1, z = -1, w = 2$ so the coordinate vector is $\begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix}$. The coordinate vectors of $\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are $\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ respectively. Putting these together we get that the representation is $\begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 \\ 1 & 2 & -1 & 1 \end{bmatrix}$.

Method 2: Suppose A is the representation with respect to S and T and B is the representation with respect to S' and T'. Then $B = Q^{-1}AP$ where P is the transition matrix from S' to S and Q is the transition matrix from T' to T. To find P, we need to find the coordinate vectors of the vectors in S' with respect to S. As S is the standard basis for P_3, the coordinate vectors are just the coefficients of the polynomials which are $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. So $P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. The matrix Q^{-1} is the transition matrix from T to T'. This is a more work to compute than P. We can either compute Q (the transition matrix from T' to T) and then find its inverse, or we can compute Q^{-1} directly by finding the coordinate vectors with respect to T' for each vector in T. We will take the second approach. The first vector in T is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so we are looking for x, y, z, w such that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + w \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$. We see that we need $x = y = z = 0, w = -1$ so the coordinate vector is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. The coordinate vectors of

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ will be $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ respectively.

This gives us that $Q^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$. Multiplying out $Q^{-1}AP$ we get the same answer as in method 1.

13. Let $L : P_2 \to P_2$ be the linear transformation given by $L(at+b) = (2a+b)t - a$.

(a) Find the representation A of L with respect to the basis $\{t+1, t-1\}$.

We'll refer to the basis $\{t+1, t-1\}$ as T. We first plug both vectors from T into L to get $L(t+1) = 3t-1$ and $L(t-1) = t-1$. We then need to find the coordinate vectors with respect to T. So we need to find x, y such that $3t - 1 = x(t+1) + y(t-1)$ so $x + y = 3, x - y = -1$ and we get that $x = 1, y = 2$ and the coordinate vector of $3t-1$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Similarly, the coordinate vector of $t-1$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Put these together to get the representation $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

(b) Find the eigenvalues and eigenvectors of A.

First take $\det(\lambda I_2 - A) = \det\left(\begin{bmatrix} \lambda - 1 & 0 \\ -2 & \lambda - 1 \end{bmatrix}\right) = (\lambda-1)^2$. So the only eigenvalue is $\lambda = 1$ with multiplicity 2. The eigenvectors will be solutions to $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -2 & 0 & 0 \end{array}\right]$. This system is $-2x = 0$ so $x = 0$ and y is anything. The eigenvectors look like $\begin{bmatrix} 0 \\ c \end{bmatrix}$ where c is a nonzero constant.

(c) Use the results from the previous part to find the eigenvalues and eigenvectors of L.

The eigenvalues of L and A are the same so the only eigenvalue of L is $\lambda = 1$. A vector $p(t) \in P_2$ will be an eigenvector of L if and only if $[p(t)]_T$ is an eigenvector of A, i.e. if $[p(t)]_T = \begin{bmatrix} 0 \\ c \end{bmatrix}$ for some nonzero constant c.

So $p(t) = 0(t+1) + c(t-1) = ct - c$ for some nonzero constant c.

14. For each matrix A, find its eigenvalues and a basis for the associated eigenspaces.

(a) $A = \begin{bmatrix} 2 & -6 & 1 \\ 0 & -1 & 0 \\ -2 & 4 & -1 \end{bmatrix}$

$\det(\lambda I_3 - A) = \det\left(\begin{bmatrix} \lambda-2 & 6 & -1 \\ 0 & \lambda+1 & 0 \\ 2 & -4 & \lambda+1 \end{bmatrix}\right) = \lambda(\lambda+1)(\lambda-1)$. The eigenvalues are $\lambda = 0$ with multiplicity 1, $\lambda = 1$ with multiplicity 1, and $\lambda = -1$ with multiplicity 1.

When $\lambda = 0$, we get the augmented matrix $\left[\begin{array}{ccc|c} -2 & 6 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -4 & -1 & 0 \end{array}\right]$. This reduces to $\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ which is the system $2x + z = 0, y = 0$. The solutions are of the form $\begin{bmatrix} x \\ 0 \\ -2x \end{bmatrix}$ which has basis $\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right\}$.

When $\lambda = 1$, we get the augmented matrix $\left[\begin{array}{ccc|c} -1 & 6 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & -4 & 2 & 0 \end{array}\right]$. The RREF is $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ which is the system $x + z = 0, y = 0$. The solutions are of the form $\begin{bmatrix} -z \\ 0 \\ z \end{bmatrix}$ which has basis $\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

When $\lambda = -1$, we get the augmented matrix $\left[\begin{array}{ccc|c} -3 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 0 \end{array}\right]$. The RREF is $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ which is the system $x - 2y = 0, z = 0$. The solutions are of the form $\begin{bmatrix} 2y \\ y \\ 0 \end{bmatrix}$ which has basis $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$.

(b) $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$

$\det(\lambda I_3 - A) = \det\left(\begin{bmatrix} \lambda-3 & 0 & 0 \\ 2 & \lambda-3 & 2 \\ -2 & 0 & \lambda-5 \end{bmatrix}\right) = (\lambda-3)^2(\lambda-5)$. The eigenvalues are $\lambda = 3$ with multiplicity 2 and $\lambda = 5$ with multiplicity 1.

When $\lambda = 3$ we get the augmented matrix $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right]$ which has RREF $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. This is the system $x + z = 0$ so y, z can be anything and $x = -z$. The solutions have the form $\begin{bmatrix} -z \\ y \\ z \end{bmatrix}$ which has basis $\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

When $\lambda = 5$ we get the augmented matrix $\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right]$ which has RREF $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. This is the system $x = 0, y + z = 0$. The solutions have the form $\begin{bmatrix} 0 \\ -z \\ z \end{bmatrix}$ which has basis $\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$.

(c) $A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$\det(\lambda I_4 - A) = \det\left(\begin{bmatrix} \lambda-4 & -2 & 0 & 0 \\ -3 & \lambda-3 & 0 & 0 \\ 0 & 0 & \lambda-2 & -5 \\ 0 & 0 & 0 & \lambda-2 \end{bmatrix}\right) = (\lambda-1)(\lambda-2)^2(\lambda-6)$. The eigenvalues are $\lambda = 1$ with multiplicity 1, $\lambda = 2$ with multiplicity 2, and $\lambda = 6$ with multiplicity 1.

When $\lambda = 1$ we get the augmented matrix $\left[\begin{array}{cccc|c} -3 & -2 & 0 & 0 & 0 \\ -3 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$. The RREF is $\left[\begin{array}{cccc|c} 1 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ which is the system $x + (2/3)y = 0, z = 0, w = 0$ so the solutions are of the form $\begin{bmatrix} -(2/3)y \\ y \\ 0 \\ 0 \end{bmatrix}$ which has basis $\left\{ \begin{bmatrix} -2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. If you don't like fractions, you could instead take the basis $\left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \end{bmatrix} \right\}$.

When $\lambda = 2$ we get the augmented matrix $\left[\begin{array}{cccc|c} -2 & -2 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. The RREF is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ which is the system $x = 0, y = 0, w = 0$ so the solutions are of the form $\begin{bmatrix} 0 \\ 0 \\ z \\ 0 \end{bmatrix}$ which has basis $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

When $\lambda = 6$ we get the augmented matrix $\left[\begin{array}{cccc|c} 2 & -2 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & -5 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$. The RREF is $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ which is the system $x - y = 0, z = 0, w = 0$

so the solutions are of the form $\begin{bmatrix} y \\ y \\ 0 \\ 0 \end{bmatrix}$ which has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

15. For each of the matrices in the previous problem, determine if A is diagonalizable. If it is diagonalizable, find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$ and find A^{100}.

    (a) This matrix is diagonalizable as it has three distinct eigenvalues (i.e. each eigenvalue has multiplicity 1). D will be the diagonal matrix with the eigenvalues of A along the diagonal and P will be the matrix whose columns are the corresponding eigenvectors. It doesn't matter what order we put the eigenvalues in (so there's more than one correct answer for D and P), but we must make sure the ordering of the eigenvectors matches the orders of the eigenvalues. $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$. $D = P^{-1}AP$ so $A = PDP^{-1}$ and $A^{100} = PD^{100}P^{-1}$. To compute A^{100} we first need to compute P^{-1}. We do this by starting with the augmented matrix $[P : I]$ and doing row operations until the left side is I and the matrix on the right will be P^{-1}. We start with $\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$. Doing the row operations $2r_1 + r_3 \to r_3$, $r_2 \leftrightarrow r_3$, $r_2 - 4r_3 \to r_2$, $r_1 + 2r_3 \to r_1$, $r_2 + r_1 \to r_1$ we get $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 2 & -4 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$ so $P^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. We then multiply everything out to get that

$$A^{100} = PD^{100}P^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^{100} \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 + 2(-1)^{100} & 1 \\ 0 & (-1)^{100} & 0 \\ -2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 0 \\ -2 & 4 & -1 \end{bmatrix}.$$

    (b) This matrix is diagonalizable. For each eigenvalue, the dimension of the eigenspace matches the multiplicity of the eigenvalue. The basis vectors for each of the eigenspaces give us three linearly independent eigenvalues. $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. To find P^{-1}, we start with $\left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$. Doing the row operations $-r_1 \to r_1, r_2 + r_3 \to r_2, r_2 - r_1 \to r_2, r_3 - r_1 \to r_3, r_1 \leftrightarrow r_3, r_2 \leftrightarrow r_3$, we get $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$ so $P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. We then multiply everything out to get that

$$A^{100} = PD^{100}P^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5^{100} & 0 & 0 \\ 0 & 3^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{100} & 0 & 0 \\ 3^{100} - 5^{100} & 3^{100} & 3^{100} - 5^{100} \\ 5^{100} - 3^{100} & 0 & 5^{100} \end{bmatrix}.$$

(c) This matrix is not diagonalizable as $\lambda = 2$ is an eigenvalue with multiplicity 2 but the eigenspace only has dimension 1.

16. Let A be an $n \times n$ matrix. Do A and A^T have the same eigenvalues? Do A and A^T have the same eigenvectors?

They have the same eigenvalues. To prove this we will show they have the same characteristic polynomial. The characteristic polynomial of A is $\det(\lambda I - A) = \det((\lambda I - A)^T) = \det((\lambda I)^T - A^T) = \det(\lambda I - A^T)$ which is the characteristic polynomial of A^T. Note that we have used that taking the transpose doesn't change the determinant and that λI is a symmetric matrix. The eigenvalues are the roots of the characteristic polynomial so as A and A^T have the same characteristic polynomials they also have the same eigenvalues.

They do not have the same eigenvectors however. Take for example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A as $A\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{x}$. However $A^T\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is not a multiple of $\mathbf{x}$. We see that $\mathbf{x}$ is an eigenvector of A but not of A^T.

17. Let A and B be $n \times n$ matrices. Suppose there exists a basis S for $\mathbb{R}^n$ such that all vectors in S are eigenvectors of both A and B. Prove that $AB = BA$.

As there are n linearly independent eigenvectors of A and of B (the vectors in S), these matrices are both diagonalizable. In fact, since they have a common basis of eigenvectors, the invertible matrix used to diagonalize them is the same. So there exists diagonal matrices D_1 and D_2 and an invertible matrix P such that $D_1 = P^{-1}AP$ and $D_2 = P^{-1}BP$. In particular, the columns of P are the vectors from S. Solving these equations for A and B we get that $A = PD_1P^{-1}$ and $B = PD_2P^{-1}$. Then $AB = PD_1P^{-1}PD_2P^{-1} = PD_1D_2P^{-1}$ and $BA = PD_2P^{-1}PD_1P^{-1} = PD_2D_1P^{-1}$. But D_1, D_2 are diagonal matrices so $D_1D_2 = D_2D_1$ and therefore $PD_1D_2P^{-1} = PD_2D_1P^{-1}$ so $AB = BA$.