Review for Final Exam

1. If A is an invertible matrix $n \times n$ matrix, which of the following must be true?

   (a) $\det(A) = 1$.

   (b) The columns of A are an orthogonal set in $\mathbb{R}^n$.

   (c) 0 is not an eigenvalue of A.

   (d) The reduced row echelon form of A is I_n.

   (e) A is diagonalizable.

   (f) The rows of A are a basis for $\mathbb{R}_n$.

2. Let $A = \begin{bmatrix} 2 & 3 & 0 & 0 & -6 \\ 0 & 0 & 0 & 1 & 5 \\ -1 & 0 & 6 & 3 & 3 \\ 0 & 1 & 4 & 2 & 0 \end{bmatrix}$.

   (a) Find the RREF of A.

   (b) What are the rank and nullity of A?

   (c) Find a basis for the null space of A.

   (d) Find a basis for the column space of A.

   (e) Find a basis for the row space of A.

   (f) Let $\mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 11 \\ 7 \end{bmatrix}$. Prove that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Find all the solutions to $A\mathbf{x} = \mathbf{b}$.

3. Let A and B be $n \times n$ matrices such that $\det(A) = 4, \det(B) = -1$.

   (a) What is $\det(A^2 B^T)$?

   (b) What are the rank and nullity of $A^2 B^T$?

   (c) Let $\mathbf{c}$ be a fixed vector in $\mathbb{R}^n$. How many solutions does $A^2 B^T \mathbf{x} = \mathbf{c}$ have? What are the solutions?

4. For what values of a is $\begin{bmatrix} a^2 \\ -3a \\ -2 \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$?

5. Determine if the following statements are true or false. Give a proof or counterexample.

   (a) If U and W are subspaces of a vector space V and $\dim U < \dim W$, then U is a subspace of W.

   (b) Any subspace of $\mathbb{R}^3$ which contains the vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ must also contain the vector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

6. Let S be the following set of vectors in $\mathbb{R}^4$.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$

   (a) Find a subset of S which is a basis for span S.

   (b) Does S contain a basis for $\mathbb{R}^4$? Is S contained in a basis for $\mathbb{R}^4$?

7. Fix a real number λ and a nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$. Determine if the following sets are subspaces of M_{nn}.

   (a) The set of all $n \times n$ matrices with eigenvalue λ.

   (b) The set of all $n \times n$ matrices with eigenvector $\mathbf{v}$.

8. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$. Determine if $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v}$ is an inner product on $\mathbb{R}^2$. Either show that it satisfies all four properties of an inner product or give an example of vectors that show it fails one of the properties.

9. Let $S = \{t^2, t, 1\}$ be the standard basis for P_2. Define an inner product on P_2 by $(p(t), q(t)) = [p(t)]_S \cdot [q(t)]_S$. Let W be the subspace of all polynomials $p(t)$ in P_2 such that $p(2) = 0$.

   (a) Find an orthogonal basis for W.

   (b) Find a basis for $W^\perp$.

10. Let A be a fixed $n \times n$ matrix. Define $L : M_{nn} \to M_{nn}$ to be $L(X) = AX - XA$.

   (a) Prove that L is a linear transformation.

   (b) Is L one-to-one? Is L onto?

11. Let $L : P_3 \to \mathbb{R}^4$ be the linear transformation $L(p(t)) = \begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \\ p'''(0) \end{bmatrix}$. Prove that L is invertible and find L^{-1}.

12. Let $L : P_3 \to M_{22}$ be the linear transformation $L(at^3 + bt^2 + ct + d) = \begin{bmatrix} a-c & 2c+d \\ b+d & 2a-b \end{bmatrix}$.

    (a) Find bases for the kernel and range of L.

    (b) Find the representation of L with respect to the bases $S = \{t^3, t^2, t, 1\}$ and $T = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

    (c) Let $S' = \{t^3, t^3 - t^2, t^3 + t^2 - t, t^3 + t^2 + t - 1\}$ and let

    $$T' = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

    Find the representation of L with respect to S' and T' two different ways: directly and using transition matrices.

13. Let $L : P_2 \to P_2$ be the linear transformation given by $L(at+b) = (2a+b)t - a$.

    (a) Find the representation A of L with respect to the basis $\{t+1, t-1\}$.

    (b) Find the eigenvalues and eigenvectors of A.

    (c) Use the results from the previous part to find the eigenvalues and eigenvectors of L.

14. For each matrix A, find its eigenvalues and a basis for the associated eigenspaces.

    (a) $A = \begin{bmatrix} 2 & -6 & 1 \\ 0 & -1 & 0 \\ -2 & 4 & -1 \end{bmatrix}$

    (b) $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$

    (c) $A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

15. For each of the matrices in the previous problem, determine if A is diagonalizable. If it is diagonalizable, find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$ and find A^{100}.

16. Let A be an $n \times n$ matrix. Do A and A^T have the same eigenvalues? Do A and A^T have the same eigenvectors?

17. Let A and B be $n \times n$ matrices. Suppose there exists a basis S for $\mathbb{R}^n$ such that all vectors in S are eigenvectors of both A and B. Prove that $AB = BA$.