Review 3 Solutions

1. Let $V = P_1$ and define $(p(t), q(t)) = p(0)q(0) + p(1)q(1)$

   (a) Prove that this is an inner product on V.

   Check the four properties of inner products.

   Property 1: $(\mathbf{u}, \mathbf{u}) \geq 0$ and $(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
   $(p(t), p(t)) = (p(0))^2 + (p(1))^2$. This is a sum of squares so it is always greater than or equal to 0. It equals 0 if and only if $p(0) = 0$ and $p(1) = 0$. As $p(t)$ is in P_1, it has the form $p(t) = at + b$. Thus $p(0) = b, p(1) = a + b$ so $p(0) = p(1) = 0$ if and only if $a = b = 0$ in which case $p(t) = 0$.
   Property 2: $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$
   $(p(t), q(t)) = p(0)q(0) + p(1)q(1) = q(0)p(0) + q(1)p(1) = (q(t), p(t))$.
   Property 3: $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$
   $(p(t)+q(t), r(t)) = (p+q)(0)r(0)+(p+q)(1)r(1) = (p(0)+q(0))r(0)+(p(1)+q(1))r(1) = (p(0)r(0) + p(1)r(1)) + (q(0)r(0) + q(1)r(1)) = (p(t), r(t)) + (q(t), r(t))$.
   Property 4: $(c\mathbf{u}, \mathbf{v}) = c(\mathbf{u}, \mathbf{v})$
   $(cp(t), q(t)) = (cp)(0)q(0)+(cp)(1)q(1) = cp(0)q(0)+cp(1)q(1) = c(p(0)q(0)+p(1)q(1)) = c(p(t), q(t))$.

   (b) Find the angle between t and 1.

   If θ is the angle between t and 1 then $\cos(\theta) = \frac{(t,1)}{\|t\|\|1\|}$. First compute the lengths. The length of a vector $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{(\mathbf{v}, \mathbf{v})}$. Then $(t, t) = (0)(0) + (1)(1) = 1$ so $\|t\| = \sqrt{1} = 1$ and $(1, 1) = (1)(1) + (1)(1) = 2$ so $\|1\| = \sqrt{2}$. Finally, $(t, 1) = (0)(1)+(1)(1) = 1$ so $\cos(\theta) = \frac{1}{\sqrt{2}}$ and $\theta = \pi/4$.

   (c) Let W be the 1-dimensional subspace of V with basis $\{t\}$. Find a basis for $W^\perp$ and $\dim W^\perp$.

   Any polynomial which is orthogonal to t will be perpendicular to all the multiples of t as well, so the elements of $W^\perp$ are exactly the polynomials in P_1 which are orthogonal to t. If $p(t)$ is in P_1, then $p(t) = at+b$ for some constants a, b. Then $(t, p(t)) = (t, at + b) = (0)(b) + (1)(a + b) = a + b$. So for $at + b$ to be orthogonal to t, it must have $a + b = 0$. Therefore $W^\perp = \{at + b | a + b = 0\} = \{at - a\} = \text{span}\{t - 1\}$. The dimension of $W^\perp$

is 1 and it has basis $\{t-1\}$.

Note that we could also compute $\dim W^{\perp}$ using the formula that $\dim W + \dim W^{\perp} = \dim P_1$ and that $\dim W = 1$ and $\dim P_1 = 2$.

(d) Find an orthonormal basis for V.

Start by finding an orthogonal basis for V. Any set of nonzero orthogonal vectors is linearly independent and the dimension of V is 2, so any two nonzero orthogonal vectors will be a basis for V. From the previous part, the set $\{t, t-1\}$ is orthogonal so this is an orthogonal basis for V. To get an orthonormal basis, divide each vector by its length. From part (a), $\|t\| = 1$. Also, $(t-1, t-1) = (-1)(-1) + (0)(0) = 1$ so $\|t-1\| = \sqrt{1} = 1$. Both t and $t-1$ are already length 1 so $\{t, t-1\}$ is an orthonormal basis.

2. Let A be an $m \times n$ matrix. If $\mathbf{v}, \mathbf{w}$ are in $\mathbb{R}^n$, define $(\mathbf{v}, \mathbf{w}) = (A\mathbf{v}) \cdot (A\mathbf{w})$. Prove that this is an inner product on $\mathbb{R}^n$ if and only if the nullity of A is 0.

The nullity of A is the dimension of the null space of A, which is the set of solutions to $A\mathbf{x} = \mathbf{0}$. The nullity of A is 0 if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. We will check the 4 properties of inner products. Properties 2-4 and the first part of property 1 are satisfied for any matrix A, however the second part of property 1 holds if and only if the nullity of A is 0.

Property 1: $(\mathbf{u}, \mathbf{u}) \geq 0$ and $(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
$(\mathbf{u}, \mathbf{u}) = (A\mathbf{u}) \cdot (A\mathbf{u})$. By properties of dot product, this is always greater than or equal to 0 and equals 0 if and only if $A\mathbf{u} = \mathbf{0}$. If A has nullity 0, then $A\mathbf{u} = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$. In this case, $(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \mathbf{0}$ so property 1 is satisfied. If A does not have nullity 0, then $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution so there exists $\mathbf{u} \neq \mathbf{0}$ with $A\mathbf{u} = \mathbf{0}$ and thus $(\mathbf{u}, \mathbf{u}) = 0$ for the nonzero vector $\mathbf{u}$ and property 1 is not satisfied.
Property 2: $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$
$(\mathbf{u}, \mathbf{v}) = (A\mathbf{u}) \cdot (A\mathbf{v}) = (A\mathbf{v}) \cdot (A\mathbf{u}) = (\mathbf{v}, \mathbf{u})$
Property 3: $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$
$(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (A(\mathbf{u} + \mathbf{v})) \cdot (A\mathbf{w}) = (A\mathbf{u} + A\mathbf{v}) \cdot (A\mathbf{w}) = ((A\mathbf{u}) \cdot (A\mathbf{w})) + ((A\mathbf{v}) \cdot (A\mathbf{w})) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$.
Property 4: $(c\mathbf{u}, \mathbf{v}) = c(\mathbf{u}, \mathbf{v})$
$(c\mathbf{u}, \mathbf{v}) = (A(c\mathbf{u})) \cdot (A\mathbf{v}) = (c(A\mathbf{u})) \cdot (A\mathbf{v}) = c((A\mathbf{u}) \cdot (A\mathbf{v})) = c(\mathbf{u}, \mathbf{v})$.

3. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space and $(\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}) = 0$, show

that $\|\mathbf{u}\| = \|\mathbf{v}\|$.

Using the properties of inner products, we can rewrite $(\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v})$ as $(\mathbf{u}, \mathbf{u}) - (\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{u}) - (\mathbf{v}, \mathbf{v}) = (\mathbf{u}, \mathbf{u}) - (\mathbf{v}, \mathbf{v})$. This is equal to 0 so $(\mathbf{u}, \mathbf{u}) - (\mathbf{v}, \mathbf{v}) = 0$ which means that $(\mathbf{u}, \mathbf{u}) = (\mathbf{v}, \mathbf{v})$. This are both nonnegative numbers so we can take square roots to get $\sqrt{(\mathbf{u}, \mathbf{u})} = \sqrt{(\mathbf{v}, \mathbf{v})}$. This equations is the same as $\|\mathbf{u}\| = \|\mathbf{v}\|$.

4. Let $V = \mathbb{R}^4$ with the dot product. Let S be the basis $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$.

   (a) Use the Gram-Schmidt process to transform S into an orthonormal basis for V.

   Label the vectors in S as $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$. We start by building an orthogonal basis which we will label $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$. The first vector is $\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$. The second is $\mathbf{v_2} = -\frac{(\mathbf{v_1}, \mathbf{u_2})}{(\mathbf{v_1}, \mathbf{v_1})}\mathbf{v_1} + \mathbf{u_2} = -\frac{6}{6}\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

   The third is $\mathbf{v_3} = -\frac{(\mathbf{v_1}, \mathbf{u_3})}{(\mathbf{v_1}, \mathbf{v_1})}\mathbf{v_1} - \frac{(\mathbf{v_2}, \mathbf{u_3})}{(\mathbf{v_2}, \mathbf{v_2})}\mathbf{v_2} + \mathbf{u_3} = -\frac{12}{6}\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. The fourth is $\mathbf{v_4} = -\frac{(\mathbf{v_1}, \mathbf{u_4})}{(\mathbf{v_1}, \mathbf{v_1})}\mathbf{v_1} - \frac{(\mathbf{v_2}, \mathbf{u_4})}{(\mathbf{v_2}, \mathbf{v_2})}\mathbf{v_2} - \frac{(\mathbf{v_3}, \mathbf{u_4})}{(\mathbf{v_3}, \mathbf{v_3})}\mathbf{v_3} + \mathbf{u_4} = -\frac{3}{6}\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$. The resulting orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$. To get an orthonormal basis, divide each vector by its length. The resulting orthonormal basis is

$$\left\{\begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\0\\-1/\sqrt{6}\end{bmatrix},\begin{bmatrix}-1/2\\1/2\\1/2\\1/2\end{bmatrix},\begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix},\begin{bmatrix}1/\sqrt{12}\\-1/\sqrt{12}\\3/\sqrt{12}\\-1/\sqrt{12}\end{bmatrix}\right\}.$$

(b) Write the vector $\begin{bmatrix}7\\-2\\1\\4\end{bmatrix}$ as a linear combination of the vectors in the basis from part (a).

If $\mathbf{v} = \begin{bmatrix}7\\-2\\1\\4\end{bmatrix} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} + a_4\mathbf{v_4}$, then $a_i = \mathbf{v_i}\cdot\mathbf{v}$. Thus $a_1 = \begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\0\\-1/\sqrt{6}\end{bmatrix}\cdot\begin{bmatrix}7\\-2\\1\\4\end{bmatrix} = -\frac{1}{\sqrt{6}}$, $a_2 = \begin{bmatrix}-1/2\\1/2\\1/2\\1/2\end{bmatrix}\cdot\begin{bmatrix}7\\-2\\1\\4\end{bmatrix} = -2$, $a_3 = \begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}\cdot\begin{bmatrix}7\\-2\\1\\4\end{bmatrix} = \frac{11}{\sqrt{2}}$, $a_4 = \begin{bmatrix}1/\sqrt{12}\\-1/\sqrt{12}\\3/\sqrt{12}\\-1/\sqrt{12}\end{bmatrix}\cdot\begin{bmatrix}7\\-2\\1\\4\end{bmatrix} = \frac{8}{\sqrt{12}}$. Therefore

$$\begin{bmatrix}7\\-2\\1\\4\end{bmatrix} = -\frac{1}{\sqrt{6}}\begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\0\\-1/\sqrt{6}\end{bmatrix} - 2\begin{bmatrix}-1/2\\1/2\\1/2\\1/2\end{bmatrix} + \frac{11}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix} + \frac{8}{\sqrt{12}}\begin{bmatrix}1/\sqrt{12}\\-1/\sqrt{12}\\3/\sqrt{12}\\-1/\sqrt{12}\end{bmatrix}.$$

5. Let $V = \mathbb{R}^4$ with the dot product. Let W be the subspace of V with basis $\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix},\begin{bmatrix}2\\1\\-1\\2\end{bmatrix}\right\}$.

(a) Find an orthogonal basis for W.

Let $\mathbf{u_1} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix}2\\1\\-1\\2\end{bmatrix}$. Using the Gram-Schmidt process, take $\mathbf{v_1} =$

$\mathbf{u_1} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$ and $\mathbf{v_2} = -\frac{(\mathbf{v_1},\mathbf{u_2})}{(\mathbf{v_1},\mathbf{v_1})}\mathbf{v_1} + \mathbf{u_2} = -\frac{3}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + \begin{bmatrix}2\\1\\-1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\-2\\1\end{bmatrix}$. The set $\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-2\\1\end{bmatrix}\right\}$ is an orthogonal basis for W.

(b) Find a basis for $W^{\perp}$.

Any vector in $\mathbb{R}^4$ which is orthogonal to both vectors in a basis for W will be orthogonal to all of W. We can therefore find the vectors in $W^{\perp}$ by finding the vectors which are orthogonal to both vectors in a basis for W. This can be done using either of the previously found bases for W. If we use the basis given in the problem, the vector $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ will be in $W^{\perp}$ if $0 = \begin{bmatrix}a\\b\\c\\d\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\\1\end{bmatrix} = a + c + d$ and $0 = \begin{bmatrix}a\\b\\c\\d\end{bmatrix} \cdot \begin{bmatrix}2\\1\\-1\\2\end{bmatrix} = 2a + b - c + 2d$. We are therefore looking for the solutions to the homogeneous system of 2 equations $a + c + d = 0, 2a + b - c + 2d = 0$. This has coefficient matrix $\begin{bmatrix}1 & 0 & 1 & 1\\2 & 1 & -1 & 2\end{bmatrix}$. The RREF of this matrix is $\begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & -3 & 0\end{bmatrix}$. It follows that c, d can be anything and $b = 3c, a = -c - d$. Therefore $W^{\perp} = \left\{\begin{bmatrix}-c-d\\3c\\c\\d\end{bmatrix}\right\} = \left\{c\begin{bmatrix}-1\\3\\1\\0\end{bmatrix} + d\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\}$. This has basis $\left\{\begin{bmatrix}-1\\3\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\}$.

(c) Find an orthogonal basis for $W^{\perp}$.

Use Gram-Schmidt to transform the basis from part (c). The steps are the same as in part (a). Note that your answer will look different depending on the order that you wrote your basis vectors in part (c). Let $\mathbf{u_1} = \begin{bmatrix}-1\\3\\1\\0\end{bmatrix}, \mathbf{u_2} = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$. Using the Gram-Schmidt process, take $\mathbf{v_1} =$

$\mathbf{u_1} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_2} = -\frac{(\mathbf{v_1},\mathbf{u_2})}{(\mathbf{v_1},\mathbf{v_1})}\mathbf{v_1} + \mathbf{u_2} = -\frac{1}{11}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Note that to avoid fractions, we can instead take a multiple of this vector, so instead we take $\mathbf{v_2} = -\begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + 11\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -3 \\ -1 \\ 11 \end{bmatrix}$. The set $\left\{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ -3 \\ -1 \\ 11 \end{bmatrix}\right\}$ is an orthogonal basis for $W^\perp$.

Note: If your basis vectors were in the other order, the orthogonal basis you would get is $\left\{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 2 \\ -1 \end{bmatrix}\right\}$.

(d) Let S be the union of the bases found in parts (a) and (c). Show that S is an orthogonal basis for $\mathbb{R}^4$.

$S = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ -3 \\ -1 \\ 11 \end{bmatrix}\right\}$. You can check that each pair of vectors is orthogonal (there are 6 pairs to check). Note that the first two vectors are orthogonal because they were part of an orthogonal basis for W, the last two vectors are orthogonal because they are part of an orthogonal basis for $W^\perp$. The other pairs are orthogonal because they will consist of one vector from W and one from $W^\perp$. S is an orthogonal set of nonzero vectors so S is linearly independent. S is a linearly independent set of size 4 in a 4-dimensional vectors space so it is a basis.

6. Let V be an inner product space with dimension 5 and let W be a 3-dimensional subspace of V.

(a) What is $\dim W^\perp$?

$\dim W^\perp = \dim V - \dim W = 2$.

(b) Suppose U is a subspace of V and $(\mathbf{u}, \mathbf{w}) = 0$ for all $\mathbf{u}$ in U and $\mathbf{w}$ in W. How are U and $W^\perp$ related?

U is a subspace of $W^\perp$.

(c) Prove that if U is a subspace of V with $\dim U = 3$ then there exist vectors $\mathbf{u}$ in U and $\mathbf{w}$ in W with $(\mathbf{u}, \mathbf{w}) \neq 0$.

If $\dim U = 3$, then U cannot be a subspace of $W^\perp$ as $W^\perp$ has dimension 2. It therefore cannot be true that $(\mathbf{u}, \mathbf{w}) = 0$ for all $\mathbf{u}$ in U and $\mathbf{w}$ in W, so there must be some $\mathbf{u}$ in U and $\mathbf{w}$ in W with $(\mathbf{u}, \mathbf{w}) \neq 0$.

7. Which of the following maps are linear transformations? For the maps which are linear transformations, find the dimension of the kernel and range.

(a) $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} ab - c \\ c + 5a \end{bmatrix}$.

This is not a linear transformation. It does not satisfy either of the properties of linear transformations. For example, $L\left(r\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = L\left(\begin{bmatrix} ra \\ rb \\ rc \end{bmatrix}\right) = \begin{bmatrix} r^2ab - rc \\ rc + 5ra \end{bmatrix}$ and $rL\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = r\begin{bmatrix} ab - c \\ c + 5a \end{bmatrix} = \begin{bmatrix} rab - c \\ rc + 5ra \end{bmatrix}$. These are not equal so the property $L(r\mathbf{v}) = rL(\mathbf{v})$ is not satisifed.

(b) $L : M_{32} \to M_{23}$ defined by $L(A) = A^T$.

This is a linear transformation. If A, B are two 3×2 matrices and r is a real number, by the properties of transpose from chapter 1 we get that $L(A + B) = (A + B)^T = A^T + B^T = L(A) + L(B)$ and $L(rA) = (rA)^T = rA^T = rL(A)$.

The kernel of L is all 3×2 matrices whose transpose is the 2×3 zero matrix. A^T is the zero matrix if and only if A is the zero matrix so the kernel of L is just the 3×2 zero matrix. The dimension of the kernel of L is 0.

The range of L is all of M_{23} since every 2×3 matrix is the transpose of some 3×2 matrix, so the dimension of the range of L is 6.

Note that L is both one-to-one and onto.

(c) $L : M_{22} \to \mathbb{R}$ defined by $L(A) = \det(A)$.

This is not a linear transformation. $L(A+B) = \det(A+B)$ and $L(A) + L(B) = \det(A) + \det(B)$ and $\det(A+B) \neq \det(A) + \det(B)$, so L does not satisfy the first property of linear transformations. It also does not satisfy the second property of linear transformations as $L(rA) = \det(rA) = r^2 \det(A) \neq r \det(A) = rL(A)$.

(d) $L : P_5 \to \mathbb{R}$ defined by $L(p(t)) = \int_0^1 p(t)\ dt$.

L is a linear transformation. If $p(t)$ and $q(t)$ are two polynomials in P_5 and r is a real number, then $L(p(t) + q(t)) = \int_0^1 p(t) + q(t)\ dt = \int_0^1 p(t)\ dt + \int_0^1 q(t)\ dt = L(p(t)) + L(q(t))$ and $L(rp(t)) = \int_0^1 rp(t)\ dt = r \int_0^1 p(t)\ dt = rL(p(t))$.

In this case it is easier to figure out the range than the kernel. The range is a subspace of $\mathbb{R}$. $\mathbb{R}$ is a 1-dimensional space so the only subspaces of $\mathbb{R}$ are the zero vector space and $\mathbb{R}$. The range is not the zero vector space since it is possible to get a nonzero number as a result of L (for example if $p(t)$ is the constant function 1 then $L(p(t)) = 1$). The range is therefore $\mathbb{R}$ so the dimension of the range is 1.

Using that $\dim \ker L + \dim \text{range } L = \dim P_5$ we get that $1 + \dim \text{range } L = 6$ so the dimension of the kernel is 5.

8. Let $L : \mathbb{R}^4 \to P_2$ be the linear transformation given by

$$L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (a-b)t^2 + (c+a)t + (b+c)$$

(a) Find a basis for the kernel of L.

The kernel of L is the subspace of $\mathbb{R}^4$ of vectors which whose image under L is $\mathbf{0}$. If $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is in the kernel of L then $a - b = 0, c + a = 0, b + c = 0$. The first two equations tell us that $b = a$ and $c = -a$ and thus the last equation $b + c = 0$ is automatically satisfied. There is no restriction on the variable d so we get that $\ker L = \left\{ \begin{bmatrix} a \\ a \\ -a \\ d \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} =$ $\operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. The two vectors are linearly independent so this is a two dimensional vector space with basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(b) Find a basis for the range of L.

The range of L is all vectors in P_2 of the form $(a-b)t^2+(c+a)t+(b+c) = a(t^2+t)+b(-t^2+1)+c(t+1)$ so the range of L is $\operatorname{span}\{t^2+t, -t^2+1, t+1\}$. These three vectors are not linearly independent as the third one is the sum of the first two so we can delete the third vector without changing the span. Therefore range $L = \operatorname{span}\{t^2+t, -t^2+1, t+1\} = \operatorname{span}\{t^2+t, -t^2+1\}$. These two vectors are linearly independent so range of L has basis $\{t^2+t, -t^2+1\}$.

Note that we found that both kernel and range of L were dimension 2. A good way to double check these dimensions is to check that they satisfy the equation $\dim \ker L + \dim \text{range } L = \dim \mathbb{R}^4$.

(c) Is L one-to-one? Onto? Invertible?

The dimension of the kernel is 2 so it is not one-to-one. The range has dimension 2 and P_2 has dimension 3 so the range is not all of P_2 and L is not onto. For L to be invertible, it must be both one-to-one and onto but it is neither so it is not invertible.

9. Let V and W be finite dimensional real vector spaces and let $L : V \to W$ be a linear transformation. Circle the correct answer to the following two multiple choice questions.

(a) If L is one-to-one, what can we say about $\dim(V)$ and $\dim(W)$?

$\dim(V) \leq \dim(W)$

L is one-to-one so $\dim\ker L = 0$ so the equation $\dim\ker L + \dim\text{range } L = \dim V$ becomes $\dim\text{range } L = \dim V$. But the range of L is a subspace of W so it has dimension less than or equal to the dimension of W, so $\dim V = \dim\text{range } L \leq \dim W$.

(b) If L is onto, what can we say about $\dim(V)$ and $\dim(W)$?

$\dim(V) \geq \dim(W)$

L is onto so $\dim\text{range } L = \dim W$ so the equation $\dim\ker L + \dim\text{range } L = \dim V$ becomes $\dim\ker L + \dim W = \dim V$ and as $\dim\ker L \geq 0$ this shows $\dim V \geq \dim W$.

10. Let $L : P_2 \to P_2$ be the map given by $L(p(t)) = tp'(t) + p(0)$

(a) Show L is a linear transformation.

We need to check the two properties of linear transformations. If $p(t), q(t)$ are vectors in P_2 then $L(p(t) + q(t)) = t(p'(t) + q'(t)) + p(0) + q(0) = (tp'(t) + p(0)) + (tq'(t) + q(0)) = L(p(t)) + L(q(t))$ so the first condition is satisfied. If $p(t)$ is a vector in P_2 and r is a real number $L(rp(t)) = trp'(t) + rp(0) = r(tp'(t) + p(0)) = rL(p(t))$ so the second condition is also satisfied.

(b) Find the matrix representing L with respect to the basis $\{t^2, t, 1\}$.

First find L evaluated at each basis element. $L(t^2) = 2t^2, L(t) = t, L(1) = 1$. The coordinate vectors of $2t^2, t, 1$ with respect to the given basis are $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ respectively. These are the columns of the matrix representing L with respect to the given basis so the matrix is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) Is L invertible? If yes, what is $L^{-1}(4t^2 - t + 3)$?

L is invertible because the matrix representing L is invertible. The matrix representing L^{-1} with respect to the basis $\{t^2, t, 1\}$ will be the inverse of the matrix in part b which is $\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The vector $4t^2 - t + 3$ has coordinate vector $\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$ so $L^{-1}(4t^2 - t + 3)$ will have coordinate vector $\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ so $L^{-1}(4t^2 - t + 3) = 2t^2 - t + 3$.

Note: This problem can also be done by first rewriting $L(p(t)) = tp'(t) + p(0)$ as $L(at^2 + bt + c) = 2at^2 + bt + c$.

11. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ 2y \\ y - 3x \end{bmatrix}$. Let S be the standard basis for $\mathbb{R}^2$ and $S' = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\}$.

Let T be the standard basis for $\mathbb{R}^3$ and $T' = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right\}$.

(a) Using the techniques of section 6.3, find the representation of L with respect to:

i. S and T

We first plug the vectors of S into L. $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. As T is the standard basis, taking the coordinate vectors with respect to T will not change these vectors so the representation is $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -3 & 1 \end{bmatrix}$.

ii. S' and T

We start by plugging the vectors in S' into L. $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$. As T is the standard basis, taking the coordinate vector with respect to T does not change the vector so the matrix we get is $\begin{bmatrix} -1 & 1 \\ 4 & -2 \\ -1 & -1 \end{bmatrix}$.

iii. S and T'

As in the first part, if we plug the vectors of S into L we get left $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$. We now need to find the coordinate vectors of each of these with respect to T'. To find the coordinate vector of $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ we need to find x, y, z such that $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. In other words, we are trying to solve the system of linear equations $x + y = 1, x + 2y = 0, y + 2z = -3$. The solution is $x = 2, y = -1, z = -1$ so the coordinate vector with respect to T' is $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. Similarly, to find the coordinate vector of $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ we're solving the linear system $x + y = -1, x + 2y = 2, y + 2z = 1$. The solution is $x = -4, y = 3, z = -1$ so the coordinate vector is $\begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix}$. Putting together these two columns we get that the representation with respect to S and T' is $\begin{bmatrix} 2 & -4 \\ -1 & 3 \\ -1 & -1 \end{bmatrix}$.

iv. S' and T'

As in the second part, if we plug the vectors of S' into L we get left $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$. We now need to find the coordinate vectors of each of these with respect to T'. To find the coordinate vector of $\begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$ we need to solve the system of linear equations $x + y = -1, x + 2y = 4, y + 2z = -1$. The solution is $x = -6, y = 5, z = -3$ so the coordinate vector with respect to T' is $\begin{bmatrix} 6 \\ 5 \\ -3 \end{bmatrix}$. Similarly, to find the coordinate vector of $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ we're solving the linear equation $x + y = 1, x + 2y = -2, y + 2z = -1$. The solution is $x = 4, y = -3, z = 1$ so the coordinate vector is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$. Putting together these two columns we get that the representation with respect to S' and T' is $\begin{bmatrix} -6 & 4 \\ 5 & -3 \\ -3 & 1 \end{bmatrix}$.

(b) Find the transition matrix

i. P from S' to S

To find the columns of P, we need to find the S coordinate vectors of each of the vectors in S'. As S is the standard basis, the coordinate vectors are the same as the original vectors so P is just the matrix with columns equal to the vectors in S'. So $P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

ii. P^{-1} from S to S'

We can either compute this by inverting P from the previous part or we can directly compute the transition matrix from S to S'. To compute this directly, we need to take each of the vectors in S and find their coordinate vectors with respect to S'. To find the coordinate vector of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with respect to S' we need to find x, y such that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. So we are solving the linear system $x = 1, 2x - y = 0$ which has solution $x = 1, y = 2$. To find the coordinate vector of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we need to solve the linear system $x = 0, 2x - y = 1$ which has solution $x = 0, y = -1$. Putting the coordinate vectors in as the columns of P^{-1} we get that $P^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$. As a check, you can verify that $PP^{-1} = I_2$. Coincidentally in this case it turns out that $P = P^{-1}$.

iii. Q from T' to T

To find Q we need to take the vectors in T' and find their coordinate vectors with respect to T. T is the standard basis so the coordinate vectors are the same as the original vectors and Q is the matrix whose columns are the vectors of T', $Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

iv. Q^{-1} from T to T'

We can either invert the matrix Q from the previous part or compute this directly by finding the T' coordinate vector of each of the vectors in T. To find the coordinate vector of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with respect to T' we need to solve the system $x + y = 1, x + 2y = 0, y + 2z = 0$. The solution is $x = 2, y = -1, z = 1/2$ so the coordinate vector is $\begin{bmatrix} 2 \\ -1 \\ 1/2 \end{bmatrix}$. To find the coordinate vector of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ with respect to T' we need to solve the system $x + y = 0, x + 2y = 1, y + 2z = 0$. The solution is $x = -1, y = 1, z = -1/2$ so the coordinate vector is $\begin{bmatrix} -1 \\ 1 \\ -1/2 \end{bmatrix}$. To find the coordinate vector of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with respect to T' we need to solve the system $x + y = 0, x + 2y = 0, y + 2z = 1$. The solution is $x = 0, y = 0, z = 1/2$ so the coordinate vector is $\begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$. Putting these together we get that $Q^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$. We can check that $QQ^{-1} = I_3$.

(c) As in section 6.5, use the representation of L with respect to S and T and the appropriate transition matrices to find the representation of L with respect to S' and T'. Check that this matches your previous answer.

Let A be the representation of L with respect to S and T and B the representation with respect to S' and T'. Then $B = Q^{-1}AP$ so we can find B using A and the transition matrices Q^{-1} and P. So $B =$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 5 & -3 \\ -3 & 1 \end{bmatrix}.$$ This matches with what we got when we computed B directly.

In fact we can also check the other two parts this way. The representation of L with respect to S' and T is AP and the representation of L with respect to S and T' is $Q^{-1}A$.

12. Determine if the statement is true or false. Prove or provide a counterexample.

(a) If $L : \mathbb{R}^6 \to M_{23}$ is a linear transformation which is onto, then L is invertible.

TRUE. L is a linear transformation between two spaces of dimension 6 hence the fact that L is onto implies it is also one-to-one. This comes from the formula $\dim \ker L + \dim \text{range } L = \dim \mathbb{R}^6$. If L is onto, the range is all of M_{23} so $\dim \text{range } L = 6$ and as $\dim \mathbb{R}^6 = 6$ we get that $\dim \ker L = 0$ so the kernel is just the zero vector space and L is one-to-one. L is both one-to-one and onto so it is invertible.

(b) Let $L : V \to W$ be a linear transformation. If $\dim W < \dim V$, then L is onto.

FALSE. For example take $V = \mathbb{R}^2, W = \mathbb{R}$ and take L to just be the zero map ($L(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in $\mathbb{R}^2$). Then $\dim W < \dim V$ and L is a linear transformation but it is not onto.

(c) If A and B are similar matrices, then $\det(A) = \det(B)$.

TRUE. A and B must be $n \times n$ matrices such that $B = P^{-1}AP$ for some invertible matrix P. Thus $\det(B) = \det(P^{-1}AP) = \det(P^{-1}) \det(A) \det(P) = (1/\det(P)) \det(A) \det(P) = \det(A)$.