Review for Exam 3

1. Let $V = P_1$ and define $(p(t), q(t)) = p(0)q(0) + p(1)q(1)$

   (a) Prove that this is an inner product on V.

   (b) Find the angle between t and 1.

   (c) Let W be the 1-dimensional subspace of V with basis $\{t\}$. Find a basis for $W^\perp$ and $\dim W^\perp$.

   (d) Find an orthonormal basis for V.

2. Let A be an $m \times n$ matrix. If $\mathbf{v}, \mathbf{w}$ are in $\mathbb{R}^n$, define $(\mathbf{v}, \mathbf{w}) = (A\mathbf{v}) \cdot (A\mathbf{w})$. Prove that this is an inner product on $\mathbb{R}^n$ if and only if the nullity of A is 0.

3. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space and $(\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}) = 0$, show that $\|\mathbf{u}\| = \|\mathbf{v}\|$.

4. Let $V = \mathbb{R}^4$ with the dot product. Let S be the basis $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$.

   (a) Use the Gram-Schmidt process to transform S into an orthonormal basis for V.

   (b) Write the vector $\begin{bmatrix} 7 \\ -2 \\ 1 \\ 4 \end{bmatrix}$ as a linear combination of the vectors in the basis from part (a).

5. Let $V = \mathbb{R}^4$ with the dot product. Let W be the subspace of V with basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

   (a) Find an orthogonal basis for W.

   (b) Find a basis for $W^\perp$.

   (c) Find an orthogonal basis for $W^\perp$.

   (d) Let S be the union of the bases found in parts (a) and (c). Show that S is an orthogonal basis for $\mathbb{R}^4$.

6. Let V be an inner product space with dimension 5 and let W be a 3-dimensional subspace of V.

   (a) What is $\dim W^{\perp}$?

   (b) Suppose U is a subspace of V and $(\mathbf{u}, \mathbf{w}) = 0$ for all $\mathbf{u}$ in U and $\mathbf{w}$ in W. How are U and $W^{\perp}$ related?

   (c) Prove that if U is a subspace of V with $\dim U = 3$ then there exist vectors $\mathbf{u}$ in U and $\mathbf{w}$ in W with $(\mathbf{u}, \mathbf{w}) \neq 0$.

7. Which of the following maps are linear transformations? For the maps which are linear transformations, find the dimension of the kernel and range.

   (a) $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} ab - c \\ c + 5a \end{bmatrix}$.

   (b) $L : M_{32} \to M_{23}$ defined by $L(A) = A^T$.

   (c) $L : M_{22} \to \mathbb{R}$ defined by $L(A) = \det(A)$.

   (d) $L : P_5 \to \mathbb{R}$ defined by $L(p(t)) = \int_0^1 p(t)\ dt$.

8. Let $L : \mathbb{R}^4 \to P_2$ be the linear transformation given by

$$L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (a - b)t^2 + (c + a)t + (b + c)$$

   (a) Find a basis for the kernel of L.

   (b) Find a basis for the range of L.

   (c) Is L one-to-one? Onto? Invertible?

9. Let V and W be finite dimensional real vector spaces and let $L : V \to W$ be a linear transformation. Circle the correct answer to the following two multiple choice questions.

   (a) If L is one-to-one, what can we say about $\dim(V)$ and $\dim(W)$?

      i. $\dim(V) \leq \dim(W)$

      ii. $\dim(V) \geq \dim(W)$

      iii. $\dim(V) = \dim(W)$

      iv. none of the above

   (b) If L is onto, what can we say about $\dim(V)$ and $\dim(W)$?

      i. $\dim(V) \leq \dim(W)$

      ii. $\dim(V) \geq \dim(W)$

iii. $\dim(V) = \dim(W)$

iv. none of the above

10. Let $L : P_2 \to P_2$ be the map given by $L(p(t)) = tp'(t) + p(0)$

(a) Show L is a linear transformation.

(b) Find the matrix representing L with respect to the basis $\{t^2, t, 1\}$.

(c) Is L invertible? If yes, what is $L^{-1}(4t^2 - t + 3)$?

11. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 2y \\ y-3x \end{bmatrix}$. Let S be the standard basis for $\mathbb{R}^2$ and $S' = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\}$.

Let T be the standard basis for $\mathbb{R}^3$ and $T' = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right\}$.

(a) Using the techniques of section 6.3, find the representation of L with respect to:

i. S and T

ii. S' and T

iii. S and T'

iv. S' and T'

(b) Find the transition matrix

i. P from S' to S

ii. P^{-1} from S to S'

iii. Q from T' to T

iv. Q^{-1} from T to T'

(c) As in section 6.5, use the representation of L with respect to S and T and the appropriate transition matrices to find the representation of L with respect to S' and T'. Check that this matches your previous answer.

12. Determine if the statement is true or false. Prove or provide a counterexample.

(a) If $L : \mathbb{R}^6 \to M_{23}$ is a linear transformation which is onto, then L is invertible.

(b) Let $L : V \to W$ be a linear transformation. If $\dim W < \dim V$, then L is onto.

(c) If A and B are similar matrices, then $\det(A) = \det(B)$.