Review for Exam 2

Note: All vector spaces are real vector spaces. Definition 4.4 will be provided on the exam as it appears in the textbook.

1. Determine if the following sets V together with operations $\oplus$ and $\odot$ are vector spaces. Either show that Definition 4.4 is satisfied or determine which properties of Definition 4.4 fail to hold.

   (a) $V = \mathbb{R}$ with $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$ and $c \odot \mathbf{u} = c + \mathbf{u}$.

   (b) $V = P_2$ with $p(t) \oplus q(t) = p'(t)q'(t)$ and $c \odot p(t) = cp(t)$.

   (c) V the set with two elements $\{\mathbf{v_1}, \mathbf{v_2}\}$ where $\mathbf{v_1} \oplus \mathbf{v_1} = \mathbf{v_2} \oplus \mathbf{v_2} = \mathbf{v_1}$ and $\mathbf{v_1} \oplus \mathbf{v_2} = \mathbf{v_2} \oplus \mathbf{v_1} = \mathbf{v_2}$ and $c \odot \mathbf{v_1} = c \odot \mathbf{v_2} = \mathbf{v_1}$.

   (d) $V = \mathbb{R}$ with $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} + 2$ and $c \odot \mathbf{u} = c(\mathbf{u} + 2) - 2$.

2. Let A be a fixed $m \times n$ matrix and let V be the set of all vectors $\mathbf{b} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ is a consistent linear system. Is V a subspace of $\mathbb{R}^m$?

3. Determine if W is a subspace of V. If it is, find a basis for W and $\dim W$.

   (a) $V = \mathbb{R}_4$, $W = \{\begin{bmatrix} a & b & c & d \end{bmatrix} | ab = cd\}$

   (b) $V = P_2$, let W be the set of all polynomials $p(t)$ in P_2 such that $p(1) = 0$.

   (c) $V = P_2$, let W be the set of all polynomials $p(t)$ in P_2 such that $p(0) = 1$.

   (d) $V = M_{22}$, let W be the set of matrices A such that $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$.

4. Let U and W be subspaces of a vector space V. Let $U + W$ be the set of all vectors in V that have the form $\mathbf{u} + \mathbf{w}$ for some $\mathbf{u}$ in U and $\mathbf{w}$ in W.

   (a) Show that $U + W$ is a subspace of V.

   (b) Show that $\dim U + W \leq \dim U + \dim W$.

5. For each set S, determine if S contains a basis for $\mathbb{R}^3$, is contained in a basis for $\mathbb{R}^3$, both, or neither.

   (a) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

   (b) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(c) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \right\}$

(d) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$

(e) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}$

6. Find a basis for span S where S is the following subset of M_{22}.

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -5 \\ 1 & 0 \end{bmatrix} \right\}$$

7. Determine if the statement is true or false. If it is true, give a proof. If it is false, find a counterexample.

   (a) If V is a nonzero vector space, then V contains infinitely many vectors.

   (b) If V has basis S and W is a subspace of V, then there exists a set T contained in S which is a basis for W.

   (c) If $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ is a set of linearly independent vectors in a vector space V and $\mathbf{w}$ is a nonzero vector in V then the set $\{\mathbf{v_1}+\mathbf{w}, \mathbf{v_2}+\mathbf{w}, ..., \mathbf{v_k}+\mathbf{w}\}$ is also linearly independent.

   (d) If two matrices have the same RREF, then they have the same row space.

   (e) If two matrices have the same RREF, then they have the same column space.

8. Let W be the following subspace of M_{23}.

$$W = \left\{ \begin{bmatrix} a & b & b-c \\ a+b & 2c & c \end{bmatrix} \right\}$$

   Find a basis for W and $\dim W$.

9. Let V be a 3-dimensional vector space with bases S and T. Let $\mathbf{v}$ be a vector such that $[\mathbf{v}]_T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $[\mathbf{v}]_S$ if $P_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$.

10. P_2 has basis $S = \{1, t, t^2 + t - 2\}$. Find a basis T for P_2 such that the transition matrix from T to S is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix}$.

11. Let $V = \mathbb{R}^4$ and let S and T be the bases $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

   (a) Find $Q_{T \leftarrow S}$ and $P_{S \leftarrow T}$.

   (b) Compute $Q_{T \leftarrow S} P_{S \leftarrow T}$.

   (c) Let $\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$. Find $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$.

   (d) Confirm that $[\mathbf{v}]_S = P_{S \leftarrow T}[\mathbf{v}]_T$ and $[\mathbf{v}]_T = Q_{T \leftarrow S}[\mathbf{v}]_S$.

12. Let A be an $n \times n$ matrix. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be a basis for $\mathbb{R}^n$ and let $T = \{A\mathbf{v_1}, A\mathbf{v_2}, ..., A\mathbf{v_n}\}$.

   (a) Prove that if A is invertible, then T is linearly independent.

   (b) Prove that for any $\mathbf{v}$ in $\mathbb{R}^n$, the n-vector $A\mathbf{v}$ is in the column space of A.

   (c) Prove that if the rank of A is less than n, then T does not span $\mathbb{R}^n$.

   (d) Use the previous parts to show that T is a basis for $\mathbb{R}^n$ if and only if A has rank n.

13. Let A be a 3×6 matrix.

   (a) What are the possible values for the rank of A?

   (b) What can you say about the nullity of A?

   (c) Suppose that the rank of A is 3. Are the rows of A linearly independent? Are the columns of A linearly independent?

14. Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ -2 & -4 & 1 & 1 \end{bmatrix}$.

   (a) Find the rank and nullity of A.

   (b) Find a basis for the row space of A.

   (c) Find a basis for the column space of A.

   (d) Find a basis for the null space of A.