Review for Exam 2 Solutions

Note: All vector spaces are real vector spaces. Definition 4.4 will be provided on the exam as it appears in the textbook.

1. Determine if the following sets V together with operations $\oplus$ and $\odot$ are vector spaces. Either show that Definition 4.4 is satisfied or determine which properties of Definition 4.4 fail to hold.

   (a) $V = \mathbb{R}$ with $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$ and $c \odot \mathbf{u} = c + \mathbf{u}$.

   V is closed under $\oplus$ and $\odot$ and satisfies properties 1-3 of the definition but fails properties 4-8.

   Closed under $\oplus$ and $\odot$: the product of any two real numbers is a real number and the sum of any two real numbers is a real number.

   Properties 1 and 2: These hold because they hold for multiplication of real numbers.

   Property 3: The number 1 plays the role of the zero vector because $\mathbf{u} \oplus 1 = \mathbf{u}$ for all $\mathbf{u} \in V$.

   Property 4: Because 1 is the zero vector, the negative of a vector $\mathbf{u}$ will be some $\mathbf{a} \in V$ such that $\mathbf{u} \oplus \mathbf{a} = 1$ so we would need $\mathbf{a} = 1/\mathbf{u}$. This property fails because 0 is in V but $1/0$ is not defined.

   Property 5: $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot (\mathbf{uv}) = c + \mathbf{uv}$ but $c \odot \mathbf{u} \oplus c \odot \mathbf{v} = (c + \mathbf{u}) \oplus (c + \mathbf{v}) = (c + \mathbf{u})(c + \mathbf{v})$ so these are not equal.

   Property 6: $(c + d) \odot \mathbf{u} = c + d + \mathbf{u}$ and $c \odot \mathbf{u} \oplus d \odot \mathbf{u} = (c + \mathbf{u})(d + \mathbf{u})$ which are not equal.

   Property 7: $c \odot (d \odot \mathbf{u}) = c + d + \mathbf{u}$ and $(cd) \odot \mathbf{u} = cd + \mathbf{u}$ which are not equal

   Property 8: $1 \odot \mathbf{u} = 1 + \mathbf{u} \neq \mathbf{u}$

   (b) $V = P_2$ with $p(t) \oplus q(t) = p'(t)q'(t)$ and $c \odot p(t) = cp(t)$.

   This is closed under $\oplus$ and $\odot$ and satisfied properties 1, 7, and 8 but fails properties 2-6.

   Closed under $\oplus$ and $\odot$: If $p(t)$ and $q(t)$ have degree at most 2, then their derivatives have degree at most 1 so the product of their derivatives has degree at most 2 and thus $p(t) \oplus q(t) = p'(t)q'(t)$ is also in P_2. Also, if $p(t)$ has degree at most 2, so does $c \odot p(t) = cp(t)$.

Property 1: $p(t) \oplus q(t) = p'(t)q'(t) = q'(t)p'(t) = q(t) \oplus p(t)$ so this is satisfied.

Property 2: $p(t) \oplus (q(t) \oplus r(t)) = p(t) \oplus q'(t)r'(t) = p'(t)(q'(t)r'(t))' = p'(t)(q'(t)r''(t) + q''(t)r'(t))$ and $(p(t) \oplus q(t)) \oplus r(t) = p'(t)q'(t) \oplus r(t) = (p'(t)q''(t) + p''(t)q'(t))r'(t)$ so these are not equal.

Property 3: There is no polynomial $e(t)$ such that $p(t) \oplus e(t) = p(t)$ for all $p(t)$. For example, take $p(t) = 1$ (the constant function). Then $p'(t) = 0$ so $p(t) \oplus e(t) = 0$ and is never equal to $p(t)$ no matter what we pick for $e(t)$. So there is no zero element and this condition fails.

Property 4: This condition automatically fails since there is no zero element.

Property 5: $c \odot (p(t) \oplus q(t)) = c \odot p'(t)q'(t) = cp'(t)q'(t)$ and $c \odot p(t) \oplus c \odot q(t) = cp(t) \oplus cq(t) = (cp(t))'(cq(t))' = c^2p'(t)q'(t)$ so these are not equal.

Property 6: $(c+d) \odot p(t) = (c+d)p(t)$ and $c \odot p(t) \oplus d \odot p(t) = cp(t) \oplus dp(t) = cd(p'(t))^2$ so these are not equal.

Property 7: $c \odot (d \odot p(t)) = c \odot dp(t) = cdp(t) = (cd) \odot p(t)$ so this condition is satisfied.

Property 8: $1 \odot p(t) = p(t)$ so this condition is satisfied.

(c) V the set with two elements $\{\mathbf{v_1}, \mathbf{v_2}\}$ where $\mathbf{v_1} \oplus \mathbf{v_1} = \mathbf{v_2} \oplus \mathbf{v_2} = \mathbf{v_1}$ and $\mathbf{v_1} \oplus \mathbf{v_2} = \mathbf{v_2} \oplus \mathbf{v_1} = \mathbf{v_2}$ and $c \odot \mathbf{v_1} = c \odot \mathbf{v_2} = \mathbf{v_1}$.

This is closed under $\oplus$ and $\odot$ and satisfies properties 1-7 of the definition but fails property 8.

Closed under $\oplus$ and $\odot$: the sum of any two elements of the set is also an element of the set and the scalar multiple is always $\mathbf{v_1}$ which is in the set.

Property 1: $\oplus$ is defined to have $\mathbf{v_1} \oplus \mathbf{v_2} = \mathbf{v_2} \oplus \mathbf{v_1}$ so this holds.

Property 2: Clearly this holds if all three vectors are the same, so we need to consider the cases where the three vectors are not all the same. If you add two $\mathbf{v_1}$'s and one $\mathbf{v_2}$ you always get $\mathbf{v_2}$, for example $(\mathbf{v_1} \oplus \mathbf{v_1}) \oplus \mathbf{v_2} = \mathbf{v_1} \oplus \mathbf{v_2} = \mathbf{v_2}$ and $\mathbf{v_1} \oplus (\mathbf{v_1} \oplus \mathbf{v_2}) = \mathbf{v_1} \oplus \mathbf{v_2} = \mathbf{v_2}$. If you add one $\mathbf{v_1}$ and two $\mathbf{v_2}$'s you always get $\mathbf{v_1}$, for example $(\mathbf{v_1} \oplus \mathbf{v_2}) \oplus \mathbf{v_2} = \mathbf{v_2} \oplus \mathbf{v_2} = \mathbf{v_1}$ and $\mathbf{v_1} \oplus (\mathbf{v_2} \oplus \mathbf{v_2}) = \mathbf{v_1} \oplus \mathbf{v_1} = \mathbf{v_1}$. This property always holds.

Property 3: The vector $\mathbf{v_1}$ is the zero vector.

Property 4: Each element is its own negative since when you add either one to itself you get $\mathbf{v_1}$ which is the zero vector.

Property 5: Scalar multiplication by c always gives you $\mathbf{v_1}$ so for any vectors $\mathbf{u}, \mathbf{v}$, $c \odot (\mathbf{u} \oplus \mathbf{v}) = \mathbf{v_1}$ and $c \odot \mathbf{u} \oplus c \odot \mathbf{v} = \mathbf{v_1} \oplus \mathbf{v_1} = \mathbf{v_1}$ so these are equal.

Property 6: $(c+d) \odot \mathbf{u} = \mathbf{v_1}$ and $c \odot \mathbf{u} \oplus d \odot \mathbf{u} = \mathbf{v_1} \oplus \mathbf{v_1} = \mathbf{v_1}$ which are equal.

Property 7: $c \odot (d \odot \mathbf{u}) = c \odot \mathbf{v_1} = \mathbf{v_1}$ and $(cd) \odot \mathbf{u} = \mathbf{v_1}$ which are equal

Property 8: $1 \odot \mathbf{u} = \mathbf{v_1}$ which is not equal to $\mathbf{u}$ if $\mathbf{u} = \mathbf{v_2}$.

(d) $V = \mathbb{R}$ with $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} + 2$ and $c \odot \mathbf{u} = c(\mathbf{u} + 2) - 2$.

This is a vector space.

Closed under $\oplus$ and $\odot$: The result of either of these operations is a real number.

Property 1: $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} + 2 = \mathbf{v} + \mathbf{u} + 2 = \mathbf{v} \oplus \mathbf{u}$ so this is satisfied.

Property 2: $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = \mathbf{u} \oplus (\mathbf{v}+\mathbf{w}+2) = \mathbf{u}+(\mathbf{v}+\mathbf{w}+2)+2 = \mathbf{u}+\mathbf{v}+\mathbf{w}+4$ and $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = (\mathbf{u}+\mathbf{v}+2) \oplus \mathbf{w} = (\mathbf{u}+\mathbf{v}+2)+\mathbf{w}+2 = \mathbf{u}+\mathbf{v}+\mathbf{w}+4$ so this is satisfied.

Property 3: The zero vector is -2 since $\mathbf{u} \oplus -2 = \mathbf{u} + (-2) + 2 = \mathbf{u}$ for all $\mathbf{u}$.

Property 4: The negative of a vector $\mathbf{u}$ is $(-1)(\mathbf{u}+4)$ since $\mathbf{u} \oplus (-1)(\mathbf{u}+4) = \mathbf{u} + (-1)(\mathbf{u}+4) + 2 = -2$ and -2 is the zero vector.

Property 5: $c \odot (\mathbf{u} \oplus \mathbf{v}) = c \odot (\mathbf{u}+\mathbf{v}+2) = c(\mathbf{u}+\mathbf{v}+2+2)-2 = c\mathbf{u}+c\mathbf{v}+4c-2$ and $c \odot \mathbf{u} \oplus c \odot \mathbf{v} = (c(\mathbf{u}+2)-2) \oplus (c(\mathbf{v}+2)-2) = (c(\mathbf{u}+2)-2) + (c(\mathbf{v}+2)-2) + 2 = c\mathbf{u} + c\mathbf{v} + 4c - 2$ so these are equal.

Property 6: $(c+d) \odot \mathbf{u} = (c+d)(\mathbf{u}+2) - 2$ and $c \odot \mathbf{u} \oplus d \odot \mathbf{u} = (c(\mathbf{u}+2) - 2) \oplus (d(\mathbf{u}+2)-2) = (c(\mathbf{u}+2)-2) + (d(\mathbf{u}+2)-2) + 2 = (c+d)(\mathbf{u}+2) - 2$ so these are equal.

Property 7: $c \odot (d \odot \mathbf{u}) = c \odot (d(\mathbf{u}+2)-2) = c(d(\mathbf{u}+2)-2+2)-2 = cd(\mathbf{u}+2)-2 = (cd) \oplus \mathbf{u}$

Property 8: $1 \odot \mathbf{u} = 1(\mathbf{u}+2) - 2 = \mathbf{u}$.

2. Let A be a fixed $m \times n$ matrix and let V be the set of all vectors $\mathbf{b} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ is a consistent linear system. Is V a subspace of $\mathbb{R}^m$?

   This is a subspace of $\mathbb{R}^m$. It is nonempty as $\mathbf{0}$ is in the set because $A\mathbf{x} = \mathbf{0}$ is consistent (it has at least the trivial solution). If $\mathbf{b}_1, \mathbf{b}_2$ are in this set then the systems $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$ both have solutions. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to these systems respectively, so $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$. We need to check if $A\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_2$ is a consistent linear system. It will be consistent as $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{b}_1 + \mathbf{b}_2$ so $\mathbf{x}_1 + \mathbf{x}_2$ is a solution. Similarly, for any real number r the system $A\mathbf{x} = r\mathbf{b}_1$ will have solution $r\mathbf{x}_1$ so $r\mathbf{b}_1$ will also be in the set.

3. Determine if W is a subspace of V. If it is, find a basis for W and $\dim W$.

(a) $V = \mathbb{R}_4$, $W = \{\begin{bmatrix} a & b & c & d \end{bmatrix} | ab = cd\}$

This is not a subspace. It is closed under multiplication but it is not closed under addition. For example, $\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$ are both in V but their sum $\begin{bmatrix} 1 & 1 & 0 & 2 \end{bmatrix}$ is not in V.

(b) $V = P_2$, let W be the set of all polynomials $p(t)$ in P_2 such that $p(1) = 0$.

This is a subspace. It is nonempty - it contains the constant zero polynomial. It is closed under addition since if $p(1) = 0$ and $q(1) = 0$ then $(p + q)(1) = p(1) + q(1) = 0$ and it is closed under scalar multiplication because if $p(1) = 0$ and r is a real number then $(rp)(1) = rp(1) = 0$.

If $p(t)$ is in W, then it looks like $p(t) = at^2 + bt + c$ and $0 = p(1) = a + b + c$. Solving for c, we get that W is all polynomials of the form $at^2 + bt - (a + b) = a(t^2 - 1) + b(t - 1)$. The set $\{t^2 - 1, t - 1\}$ is therefore a spanning set for W and it is a linearly independent set so it is a basis. The dimension of the space is 2.

Note: Some other correct bases for this subspace are $\{-t^2 + t, -t^2 + 1\}$ and $\{t^2 - t, -t + 1\}$. These are the bases you'd get by solving $a + b + c = 0$ for a or b.

(c) $V = P_2$, let W be the set of all polynomials $p(t)$ in P_2 such that $p(0) = 1$.

This is not a subspace. These are polynomials with constant term 1. This set is not closed under addition since the sum of any two polynomials with constant term 1 has constant term 2 and it is not closed under scalar multiplication because if $p(0) = 1$ then $(rp)(0) = rp(0) = r$.

(d) $V = M_{22}$, let W be the set of matrices A such that $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$.

This is a subspace. Denote the all ones matrix as $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The set W is nonempty as it contains the identity matrix and the zero matrix and J itself. If A_1, A_2 are in W then $A_1 J = JA_1$ and $A_2 J = JA_2$ so $(A_1 + A_2)J = A_1 J + A_2 J = JA_1 + JA_2 = J(A_1 + A_2)$ so $A_1 + A_2$ is also in W (this shows W is closed under addition). If A is in W and r is a real number then $(rA)J = rAJ = rJA = J(rA)$ so rA is also in W (this shows W is closed under scalar multiplication).

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix. Then $JA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + c & b + d \\ a + c & b + d \end{bmatrix}$ and $AJ = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a + b & a + b \\ c + d & c + d \end{bmatrix}$. The matrix A

is in W if and only if $AJ = JA$ which happens if and only if $a = d$ and $b = c$. The set W is therefore all 2×2 matrices of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ spans W and is linearly independent so it is a basis for W. The dimension of W is 2.

4. Let U and W be subspaces of a vector space V. Let $U + W$ be the set of all vectors in V that have the form $\mathbf{u} + \mathbf{w}$ for some $\mathbf{u}$ in U and $\mathbf{w}$ in W.

   (a) Show that $U + W$ is a subspace of V.

   The set $U + W$ is nonempty - in fact it contains both U and W since both spaces contain $\mathbf{0}$. To check if $U + W$ is closed under addition, take $\mathbf{v_1}, \mathbf{v_2}$ to be any vectors in $U + W$. They can be written as $\mathbf{v_1} = \mathbf{u_1} + \mathbf{w_1}$ and $\mathbf{v_2} = \mathbf{u_2} + \mathbf{w_2}$ for some $\mathbf{u_1}, \mathbf{u_2}$ in U and $\mathbf{w_1}, \mathbf{w_2}$ in W. Their sum is equal to $\mathbf{v_1} + \mathbf{v_2} = \mathbf{u_1} + \mathbf{w_1} + \mathbf{u_2} + \mathbf{w_2} = (\mathbf{u_1} + \mathbf{u_2}) + (\mathbf{w_1} + \mathbf{w_2})$. U and W are subspaces so they are closed under addition so $\mathbf{u_1} + \mathbf{u_2}$ is in U and $\mathbf{w_1} + \mathbf{w_2}$ is in W. We have therefore written $\mathbf{v_1} + \mathbf{v_2}$ as the sum of a vector in U and a vector in W so $\mathbf{v_1} + \mathbf{v_2}$ is in $U + W$. This shows that $U + W$ is closed under addition. Next check that $U + W$ is closed under scalar multiplication. If $\mathbf{v}$ is in $U + W$ then $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for some $\mathbf{u}$ in U and $\mathbf{w}$ in W. If r is a real number, then $r\mathbf{v} = r\mathbf{u} + r\mathbf{w}$. U and W are closed under scalar multiplication so $r\mathbf{u}$ is in U and $r\mathbf{w}$ is in W. We have therefore written $r\mathbf{v}$ as the sum of a vector in U and a vector in W so $r\mathbf{v}$ is in $U+W$ and $U+W$ is closed under scalar multiplication.

   (b) Show that $\dim U + W \leq \dim U + \dim W$.

   Suppose $\dim U = n$ and $\dim W = m$. Let $S = \{\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_n}\}$ be a basis for U and let $T = \{\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_m}\}$. Take R to be the set $R = \{\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_n}, \mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_m}\}$. We will show that R is a spanning set for $U+W$. Any vector in $U+W$ can be written in the from $\mathbf{u}+\mathbf{w}$ for some $\mathbf{u}$ in U and $\mathbf{w}$ in W. As S is a basis for U, we can write $\mathbf{u}$ as a linear combination of the $\mathbf{u_i}$ and similarly we can write $\mathbf{w}$ as a linear combination of the $\mathbf{w_i}$ so any vector in $U + W$ can be written as a linear combination of the vectors in R. It follows that R is a spanning set for $U + W$ so R contains a basis for $U+W$. The size of R is $n+m$ so $\dim U+W \leq n+m = \dim U+\dim W$.

   Note: The above proof assumes that U and W have finite bases, but the statement is true even if the spaces are the zero vector space or infinite dimensional. If one of the spaces was the zero vector space, say U, then

$U+W$ would be equal to W and the statement $\dim U+W \leq \dim U+\dim W$ would become $\dim W \leq 0+\dim W$ which is clearly true. If one or both of the spaces was infinite dimensional, then $U+W$ would also have to be infinite dimensional since U and W are both subspaces of $U+W$. (Recall that the subspaces of finite dimensional vector spaces are all finite dimensional of dimension less than or equal to the original space, see hw 7).

5. For each set S, determine if S contains a basis for $\mathbb{R}^3$, is contained in a basis for $\mathbb{R}^3$, both, or neither.

   (a) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

   Contained in a basis. It is linearly independent so it must be contained in a basis and it is too small to contain a basis.

   (b) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

   Contains a basis. This set spans $\mathbb{R}^3$ so it contains a basis but it is too big to be contained in a basis.

   (c) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \right\}$

   Both. This set is both linearly independent and spans $\mathbb{R}^3$ so it is a basis and therefore both contains and is contained in a basis.

   (d) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$

   Neither. This set is not linearly independent (the last vector is the second minus the first) and its span has dimension 2 so is not all of $\mathbb{R}^3$. Since it is not linearly independent it cannot be contained in a basis and it does not span so it cannot contain a basis.

   (e) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}$

   Neither. This set is not linearly independent and does not span $\mathbb{R}^3$.

6. Find a basis for span S where S is the following subset of M_{22}.

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -5 \\ 1 & 0 \end{bmatrix} \right\}$$

We take a linear combination and set it equal to $\mathbf{0}$ as follows:

$$a \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} + c \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} + e \begin{bmatrix} -1 & -5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is the same as $\begin{bmatrix} a+c+d-e & a+b+2c-5e \\ 2b+2c+d+e & -a+b-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which gives us the linear system $a+c+d-e = 0, a+b+2c-5e = 0, 2b+2c+d+e = 0, -a+b-d = 0$. This has coefficient matrix $\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 & -5 \\ 0 & 2 & 2 & 1 & 1 \\ -1 & 1 & 0 & -1 & 0 \end{bmatrix}$. Doing row operations gets REF of $\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The leading ones are in columns 1,2, and 4 so we can take the basis to be the first, second, and fourth elements of S. The basis we get is $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right\}$.

7. Determine if the statement is true or false. If it is true, give a proof. If it is false, find a counterexample.

   (a) If V is a nonzero vector space, then V contains infinitely many vectors.

   True. If V is nonzero, it contains a vector $\mathbf{v} \neq \mathbf{0}$. V must be closed under scalar multiplication so V also contains all scalar multiples $r\mathbf{v}$ where r is a real number. If any two scalar multiples $r\mathbf{v}$ and $s\mathbf{v}$ are equal, then $r\mathbf{v} = s\mathbf{v}$ so $(r-s)\mathbf{v} = \mathbf{0}$ and as $\mathbf{v} \neq \mathbf{0}$, this forces $r = s$. This shows that the scalar multiples of $\mathbf{v}$ are all distinct. There are infinitely many real numbers, so there are also infinitely many scalar multiples of $\mathbf{v}$ and V must contain infinitely vectors. Note that V may have a finite basis, but V itself must contain infinitely many vectors.

   (b) If V has basis S and W is a subspace of V, then there exists a set T contained in S which is a basis for W.

   False. For example take $V = \mathbb{R}^2$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $W = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \right\}$. Then S is a basis for V and W is a subspace of V. Note that W is 1-dimensional so any basis for W consists of exactly one vector. However S cannot contain a basis for W since the vectors in S are not in W.

(c) If $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ is a set of linearly independent vectors in a vector space V and $\mathbf{w}$ is a nonzero vector in V then the set $\{\mathbf{v_1} + \mathbf{w}, \mathbf{v_2} + \mathbf{w}, ..., \mathbf{v_k} + \mathbf{w}\}$ is also linearly independent.

False. For example, we could take $\mathbf{w} = -\mathbf{v_1}$. This is nonzero since $\mathbf{v_1}$ is in S which is linearly independent, but the new set will contain $\mathbf{0}$ so it will be linearly dependent.

(d) If two matrices have the same RREF, then they have the same row space.

True. Any two matrices with the same RREF must be row equivalent and any two row equivalent matrices have the same row space (see section 4.9).

(e) If two matrices have the same RREF, then they have the same column space.

False. For example, the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ both have RREF $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ but their column spaces are span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ which are not the same.

8. Let W be the following subspace of M_{23}.

$$W = \left\{ \begin{bmatrix} a & b & b-c \\ a+b & 2c & c \end{bmatrix} \right\}$$

Find a basis for W and $\dim W$.

We can rewrite W as $W = \left\{ a \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \right\} =$ span $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \right\}$. These three elements span W. They are also linearly independent, so we get that $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \right\}$ is a basis for W. This set has size 3, so $\dim W = 3$.

9. Let V be a 3-dimensional vector space with bases S and T. Let $\mathbf{v}$ be a vector such that $[\mathbf{v}]_T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find $[\mathbf{v}]_S$ if $P_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$.

Using the formula that $[\mathbf{v}]_S = P_{S \leftarrow T}[\mathbf{v}]_T$ we get that $[\mathbf{v}]_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} =$

$$\begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}.$$

10. P_2 has basis $S = \{1, t, t^2 + t - 2\}$. Find a basis T for P_2 such that the transition matrix from T to S is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix}$.

    Let $T = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$. The transition matrix has i-th column equal to $[\mathbf{w}_i]_S$. Hence $[\mathbf{w}_1]_S = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ so $\mathbf{w}_1 = 1(1) + 0(t) + 1(t^2 + t - 2) = t^2 + t - 1$. Similarly $\mathbf{w}_2 = 2(1) + 1(t) + 3(t^2 + t - 2) = 3t^2 + 4t - 4$ and $\mathbf{w}_3 = 0(1) + 0(t) - 1(t^2 + t - 2) = -t^2 - t + 2$ so $T = \{t^2 + t - 1, 3t^2 + 4t - 4, -t^2 - t + 2\}$.

11. Let $V = \mathbb{R}^4$ and let S and T be the bases $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

    (a) Find $Q_{T \leftarrow S}$ and $P_{S \leftarrow T}$.

    Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$. We can find $P_{S \leftarrow T}$ by taking the matrix $[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4 : \mathbf{w}_1\mathbf{w}_2\mathbf{w}_3\mathbf{w}_4]$ and doing row operations to get the identity on the left. So we have $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \end{array}\right]$. The row operations $(1/2)r_2 \to r_2, (1/3)r_3 \to r_3, (1/4)r_4 \to r_4$ give us the matrix $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ so $P_{S \leftarrow T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

    We do a similar process to get $Q_{T \leftarrow S}$. Start with $\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \end{array}\right]$. Do the row operations $(1/2)r_2 \to r_2, (1/3)r_3 \to r_3, (1/4)r_4 \to r_4$ then $r_1 -$

$r_2 \to r_1, r_2 - r_3 \to r_2, r_3 - r_4 \to r_4$ to get $\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

so $Q_{T\leftarrow S} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) Compute $Q_{T\leftarrow S}P_{S\leftarrow T}$.

These are inverses, so their product is I_4, the 4×4 identity.

(c) Let $\mathbf{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$. Find $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$.

The vector $[\mathbf{v}]_S$ will be the solution to the system $\left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 4 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right]$ so

$[\mathbf{v}]_S = \begin{bmatrix} 4 \\ 2 \\ 4/3 \\ 1 \end{bmatrix}$.

The vector $[\mathbf{v}]_T$ will be the solution to the system $\left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 4 \\ 0 & 2 & 2 & 2 & 4 \\ 0 & 0 & 3 & 3 & 4 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right]$ so

$[\mathbf{v}]_T = \begin{bmatrix} 2 \\ 2/3 \\ 1/3 \\ 1 \end{bmatrix}$.

(d) Confirm that $[\mathbf{v}]_S = P_{S\leftarrow T}[\mathbf{v}]_T$ and $[\mathbf{v}]_T = Q_{T\leftarrow S}[\mathbf{v}]_S$.

$$P_{S\leftarrow T}[\mathbf{v}]_T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2/3 \\ 1/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4/3 \\ 1 \end{bmatrix} = [\mathbf{v}]_S$$

$$Q_{T\leftarrow S}[\mathbf{v}]_S = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 4/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2/3 \\ 1/3 \\ 1 \end{bmatrix} = [\mathbf{v}]_T$$

12. Let A be an $n \times n$ matrix. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be a basis for $\mathbb{R}^n$ and let $T = \{A\mathbf{v_1}, A\mathbf{v_2}, ..., A\mathbf{v_n}\}$.

(a) Prove that if A is invertible, then T is linearly independent.

To show that T is linearly independent, take a linear combination of the vectors in T and set it equal to $\mathbf{0}$. This gives us the equation $a_1A\mathbf{v_1} + a_2A\mathbf{v_2} + ... + a_nA\mathbf{v_n} = \mathbf{0}$ which is the same as $A(a_1\mathbf{v_1}) + A(a_2\mathbf{v_2}) + ... + A(a_n\mathbf{v_n}) = \mathbf{0}$. We need to prove that all the a_i must equal 0. As A is invertible, we can multiply both sides of this equation by A^{-1} on the left to get $A^{-1}A(a_1\mathbf{v_1}) + A^{-1}A(a_2\mathbf{v_2}) + ... + A^{-1}A(a_n\mathbf{v_n}) = A^{-1}\mathbf{0}$ which simplifies to $a_1\mathbf{v_1} + a_2\mathbf{v_2} + ... + a_n\mathbf{v_n} = \mathbf{0}$. By the linear independence of S, all a_i must equal 0.

(b) Prove that for any $\mathbf{v}$ in $\mathbb{R}^n$, the n-vector $A\mathbf{v}$ is in the column space of A.

Let $\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and denote the columns of A as $\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_n}$. Then $A\mathbf{v} = A\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1\mathbf{c_1} + a_2\mathbf{c_2} + ... + a_n\mathbf{c_n}$. We see that $A\mathbf{v}$ is a linear combination of the columns of A so it is in the column space of A.

(c) Prove that if the rank of A is less than n, then T does not span $\mathbb{R}^n$.

By part (b), all the vectors in T are contained in the column space of A and hence span T is contained in the column space of A. If the rank of A is less than n, then the dimension of the column space of A is less than n. As span T is contained in the column space of A, it also has dimension less than n so span T cannot be all of $\mathbb{R}^n$.

(d) Use the previous parts to show that T is a basis for $\mathbb{R}^n$ if and only if A has rank n.

If A has rank n, then A is invertible. By part (a), T is linearly independent and it consists of n vectors in $\mathbb{R}^n$ so T must be a basis for $\mathbb{R}^n$. If A does not have rank n, then it must have rank less than n. By part (b), T would not span so T would not be a basis.

13. Let A be a 3×6 matrix.

(a) What are the possible values for the rank of A?

The rank can be 0, 1, 2, 3. It cannot be any larger because the dimension of the row space cannot be larger than 3 since there are three rows.

(b) What can you say about the nullity of A?

The nullity is equal to the number of columns, which is 6, minus the rank. The possible values for rank are 0, 1, 2, 3 so the possible values for the nullity are 6, 5, 4, 3.

(c) Suppose that the rank of A is 3. Are the rows of A linearly independent? Are the columns of A linearly independent?

If the rank is 3 then both the row and column spaces have dimension 3. There are 3 rows and their span (the row space) is dimension 3 so they must be linearly independent. There are 6 columns and the dimension of their span is 3 so they are not linearly independent.

14. Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ -2 & -4 & 1 & 1 \end{bmatrix}$.

(a) Find the rank and nullity of A.

All four parts of this problem can be done by first finding REF or RREF of A. Doing the row operations $r_3 + 2r_1 \to r_3, r_3 - r_2 \to r_3$ gives us the REF of A which is $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. In this case it is also the RREF of A. There are two columns containing leading ones (1 and 3) so the rank is 2. There are two columns which do not contain leading ones (2 and 4) so the nullity is 2.

(b) Find a basis for the row space of A.

The nonzero rows in REF of A are a basis for the row space of A so a basis is $\left\{\begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 \end{bmatrix}\right\}$.

(c) Find a basis for the column space of A.

The column vectors of A corresponding to the columns of REF with leading ones (1 and 3) are a basis for the column space so the first and third columns of A are a basis for the column space of A. The basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(d) Find a basis for the null space of A.

The null space is the set of solutions to $A\mathbf{x} = \mathbf{0}$. If we use the variables x, y, z, w then we see from the REF of A that columns 2 and 4 do not contain leading ones so the variables y, w can be any real numbers and $z = -3w, x = -2y - w$ so the null space is all vectors of the form $\begin{bmatrix} -2y - w \\ y \\ -3w \\ w \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$. A basis for this space is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$.