Review for Exam 1

1. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$.
   Compute $D = AB^T + 2C^2$. Which of the following terms describe D: diagonal, scalar, upper triangular, lower triangular, symmetric, skew symmetric, invertible.
   Circle all (if any) that apply.

2. Determine if each statement is true or false. If it is true give a proof. If it is false find a counterexample.

   (a) If A is a scalar $n \times n$ matrix, then $AB = BA$ for all $n \times n$ matrices B.

   (b) If A is an $n \times n$ matrix and $A^k = I_n$ for some positive integer k, then A is invertible.

   (c) If A is an invertible $n \times n$ matrix then $A^k = I_n$ for some positive integer k.

   (d) An upper triangular matrix A is invertible if and only if all the entries on the diagonal of A are nonzero.

   (e) If A is an $n \times n$ matrix with $\det(A) = 3$, then $\det(A^2 - A) = 6$.

3. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Determine if A is a linear combination of the matrices B, C, D.

   (a) $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

4. Let A and B be $n \times n$ matrices such that $B^T A^2 = I$.

   (a) Show that A is invertible and find A^{-1}.

   (b) Determine if $A^T B - A^{-1}$ is symmetric, skew symmetric, or neither.

5. The product of any two upper triangular $n \times n$ matrices is upper triangular. Prove this fact for 3×3 matrices.

6. Let $A^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find all solutions to the linear system $A^2\mathbf{x} = \mathbf{b}$.

7. Find all a for which the linear system

$$x + y - z = 2$$

$$x + 2y + z = 3$$

$$x + y + (a^2 - 5)z = a$$

has no solutions, one solution, and infinitely many solutions.

8. Find the augmented matrix of each system of linear equations. Use Gaussian elimination or Gauss-Jordan reduction to solve the linear system.

   (a) $y + 3z = -10$
   $x + 2z = 11$
   $2x - y + 7z = 14$

   (b) $x + 3y - z + w = 5$
   $x - 6y + 2z = 1$
   $2x + w = 6$

   (c) $2x + 3y + z - w = 1$
   $x - y + w = 2$
   $4x + y + z + w = 4$
   $6x + 3y - 7z - w = 12$

9. Find the inverse of A or show that A is not invertible.

   (a) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \\ 3 & 4 & 5 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 7 & 5 \\ 3 & -1 & 2 \\ 5 & 13 & 12 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

10. Let A be an $n \times n$ matrix such that the n-th row is a linear combination of rows 1 through $n - 1$. Prove that A is not invertible.

11. For what value or values of k is the matrix $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & -2 & 1 \\ 2 & 6 & k \end{bmatrix}$ invertible?

12. Let A be a 4×4 matrix with $\det(A) = -4$. What is the RREF of A? How many solutions does the homogeneous system $A\mathbf{x} = \mathbf{0}$ have?

13. Suppose A is a 3×3 matrix with $\det(A) = 6$. Compute the determinant of the following matrices.

    (a) A^3

    (b) $2A$

    (c) $(A^T)^{-1}$

14. Suppose A and B are invertible 3×3 matrices and $AB^T = 2B^2$. If $\det(A) = 5$, what is $\det(B)$?

15. Compute the determinant of A.

    (a) $A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$

    (b) $A = \begin{bmatrix} 0 & 1 & -2 \\ 5 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$

    (c) $A = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 3 & 4 & 7 \\ 0 & 0 & 5 & 8 \\ 2 & 0 & 0 & 9 \end{bmatrix}$

16. The matrix $A = \begin{bmatrix} 1 & 2 & 6 & 8 \\ 1 & 3 & 0 & 9 \\ 1 & 4 & 0 & 10 \\ 1 & 5 & 7 & 0 \end{bmatrix}$ is invertible. Find all solutions to the following linear systems.

    (a) $A^{-1}\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

    (b) $A\mathbf{x} = \mathbf{0}$