Review for Exam 1

1. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$.
   Compute $D = AB^T + 2C^2$. Which of the following terms describe D: diagonal, scalar, upper triangular, lower triangular, symmetric, skew symmetric, invertible.
   Circle all (if any) that apply.
   $AB^T = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix}$ and $C^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so $D = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}$.
   D is symmetric ($D^T = D$) and invertible ($\det(D) = -25 \neq 0$). It is not any of the other things.

2. Determine if each statement is true or false. If it is true give a proof. If it is false find a counterexample.
   (a) If A is a scalar $n \times n$ matrix, then $AB = BA$ for all $n \times n$ matrices B.
   TRUE. If A is a scalar matrix $n \times n$ matrix, then $A = kI$ for some constant k. Then $AB = (kI)B = k(IB) = kB$ and $BA = B(kI) = k(BI) = kB$ so $AB = BA$.
   (b) If A is an $n \times n$ matrix and $A^k = I_n$ for some positive integer k, then A is invertible.
   TRUE. The matrix A^{k-1} is the inverse of A. Note that A^0 is defined to be I_n, so this makes sense even if $k = 1$.
   (c) If A is an invertible $n \times n$ matrix then $A^k = I_n$ for some positive integer k.
   FALSE. The scalar matrix $2I_n$ is invertible as it has inverse $\frac{1}{2}I_n$. However $(2I_n)^k = 2^k I_n$ and $2^k \neq 1$ for any $k > 0$ so no power of $2I_n$ is equal to I_n.
   (d) An upper triangular matrix A is invertible if and only if all the entries on the diagonal of A are nonzero.
   TRUE. A is invertible if and only if $\det(A) \neq 0$. We proved in class that the determinant of an upper triangular matrix is the product of the diagonal entries, so $\det(A) \neq 0$ if and only if the diagonal entries are all nonzero.
   (e) If A is an $n \times n$ matrix with $\det(A) = 3$, then $\det(A^2 - A) = 6$.
   FALSE. A counterexample would be $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$. This matrix has determinant 3 but $A^2 - A = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$ which has determinant 0, not 6.

3. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Determine if A is a linear combination of the matrices B, C, D.

   (a) $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}$

   A linear combination of B, C, D is any expression of the form $bB+cC+dD$ where b, c, d are constants. To see if A is a linear combination of B, C, D we need to see if we can find constants b, c, d such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = b\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} b & 2c \\ d & 4d \end{bmatrix}$. So we want to see if the system $1 = b, 2 = 2c, 3 = d, 4 = 4d$. There are no solutions because d cannot be both 3 and 1 so A cannot be written as a linear combination of B, C, and D.

   (b) $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

   We want to find b, c, d such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b+c-d & b \\ c & d \end{bmatrix}$. This gives us the system $b + c - d = 1, b = 2, c = 3, d = 4$ which has solution $b = 2, c = 3, d = 4$ so

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + 4\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

4. Let A and B be $n \times n$ matrices such that $B^T A^2 = I$.

   (a) Show that A is invertible and find A^{-1}.

   $I = B^T A^2 = (B^T A)A$ so $A^{-1} = B^T A$.

   (b) Determine if $A^T B - A^{-1}$ is symmetric, skew symmetric, or neither.

   By part (a), we can rewrite $A^T B - A^{-1}$ as $A^T B - B^T A$. To check if this matrix is symmetric, skew symmetric, or neither, take the transpose of $A^T B - B^T A$ and see if we get back $A^T B - B^T A$, $-(A^T B - B^T A)$ or neither. The transpose is $(A^T B - B^T A)^T = (A^T B)^T - (B^T A)^T = B^T (A^T)^T - A^T (B^T)^T = B^T A - A^T B = -(A^T B - B^T A)$ so it is skew symmetric.

5. The product of any two upper triangular $n \times n$ matrices is upper triangular. Prove this fact for 3×3 matrices.

   Let A and B be 3×3 upper triangular matrices. Then they look like $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$. To show that AB is upper triangular, we just need to show that the $(2,1)$, $(3,1)$ and $(3,2)$ entries of AB

are 0. The the $(2,1)$ entry is $(0)(b_{11}) + (a_{22})(0) + (a_{23})(0) = 0$, the $(3,1)$ entry is $(0)(b_{11}) + (0)(0) + (a_{33})(0) = 0$ and the $(3,2)$ entry is $(0)(b_{12}) + (0)(b_{22}) + (a_{33})(0) = 0$. The entries below the diagonal of AB are all 0 so AB is upper triangular.

6. Let $A^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find all solutions to the linear system $A^2\mathbf{x} = \mathbf{b}$.

   Given the equation $A^2\mathbf{x} = \mathbf{b}$, we can multiply both sides by A^{-1} on the left to get $A\mathbf{x} = A^{-1}\mathbf{b}$ and doing this again we get $\mathbf{x} = (A^{-1})^2\mathbf{b}$ so the only solution is $(A^{-1})^2\mathbf{b}$. $(A^{-1})^2 = \begin{bmatrix} -1 & 2 \\ -1 & -2 \end{bmatrix}$ so $\mathbf{x} = (A^{-1})^2\mathbf{b} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$.

7. Find all a for which the linear system

$$x + y - z = 2$$

$$x + 2y + z = 3$$

$$x + y + (a^2 - 5)z = a$$

   has no solutions, one solution, and infinitely many solutions.

   The augmented matrix for this system is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right]$. Doing the elementary row operations $r_2 - r_1 \to r_2$ and $r_3 - r_1 \to r_3$, this becomes $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$. To get the matrix in row echelon form, we want to divide row three by $a^2 - 4$, but we can only do this if $a^2 - 4 \neq 0$ so we need to consider the cases where $a^2 - 4 = 0$ and $a^2 - 4 \neq 0$ separately. The case where $a^2 - 4 = 0$ splits into two cases, $a = 2$ and $a = -2$ so we have three cases to consider.

   If $a^2 - 4 \neq 0$ then the row echelon form of the matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{a-2}{a^2-4} \end{array}\right]$. This gives us the equations $z = \frac{a-2}{a^2-4}$, $y + 2z = 1$ and $x + y - z = 2$. We see that in this case there is one solution, $z = \frac{a-2}{a^2-4}, y = 1 - \frac{2(a-2)}{a^2-4}, x = 2 - (1 - \frac{2(a-2)}{a^2-4}) + \frac{a-2}{a^2-4}$.

   If $a = 2$ then the matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so the equations are $x + y - z = 2, y + 2z = 1, 0 = 0$. There are infinitely many solutions as z can be anything.

If $a = -2$ then the matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right]$ and the last equation is $0 = -4$ which has no solutions.

Hence the system has no solutions when $a = -2$, one solution when $a \neq 2, -2$, and infinitely many solutions when $a = 2$.

8. Find the augmented matrix of each system of linear equations. Use Gaussian elimination or Gauss-Jordan reduction to solve the linear system.

   (a) $y + 3z = -10$
   $x + 2z = 11$
   $2x - y + 7z = 14$

   The augmented matrix is $\left[\begin{array}{ccc|c} 0 & 1 & 3 & -10 \\ 1 & 0 & 2 & 11 \\ 2 & -1 & 7 & 14 \end{array}\right]$. This can be gotten into row echelon form by doing the following row operations: $r_1 \leftrightarrow r_2$, $-2r_1 + r_3 \to r_3$, $r_3 + r_2 \to r_3$, $\frac{1}{6}r_3 \to r_3$. The resulting matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 11 \\ 0 & 1 & 3 & -10 \\ 0 & 0 & 1 & -3 \end{array}\right]$.

   We then proceed one of two different ways. One way to finish solving the problem is to write this as the equations $x+2z = 11, y+3z = -10, z = -3$. Using back substitution we get that $y = -10 + 9 = -1, x = 11 + 6 = 17$ so the solution is $\begin{bmatrix} 17 \\ -1 \\ -3 \end{bmatrix}$.

   Another way to finish the problem is to do row operations to get the matrix in reduced row echelon form. The following row operations will get the matrix into reduced row echelon form: $-3r_3 + r_2 \to r_2$, $-2r_3 + r_1 \to r_1$ and the resulting matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right]$ and the solution can be read off from here as $\begin{bmatrix} 17 \\ -1 \\ -3 \end{bmatrix}$.

   (b) $x + 3y - z + w = 5$
   $x - 6y + 2z = 1$
   $2x + w = 6$

   The augmented matrix is $\left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 5 \\ 1 & -6 & 2 & 0 & 1 \\ 2 & 0 & 0 & 1 & 6 \end{array}\right]$. The following elementary row operations will get a matrix in row echelon form: $-r_1 + r_2 \to r_2$,

$-2r_1 + r_3 \to r_3$, $-\frac{1}{9}r_2 \to r_2$, $6r_2 + r_3 \to r_3$, $-3r_3 \to r_3$. The resulting matrix is $\left[\begin{array}{cccc:c} 1 & 3 & -1 & 1 & 5 \\ 0 & 1 & -1/3 & 1/9 & 4/9 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$.

We see from here that $w = 4$. There is no leading one in the z column so z can be anything. To find y, use that $y - \frac{1}{3}z + \frac{1}{9}w = \frac{4}{9}$ so $y = \frac{1}{3}z$. Then $x + 3y - z + w = 5$ so $x - 3y + 3y + 4 = 5$ and $x = 1$. There are infinitely many solutions, they are $\begin{bmatrix} 1 \\ \frac{1}{3}z \\ z \\ 4 \end{bmatrix}$ where z is anything. This could also be written as $\begin{bmatrix} 1 \\ y \\ 3y \\ 4 \end{bmatrix}$.

Alternatively, we could do the row operations $-\frac{1}{9}r_3 + r_2 \to r_2, -r_3 + r_1 \to r_1, -3r_2 + r_1 \to r_1$ to get the matrix in reduced row echelon form. The resulting matrix is $\left[\begin{array}{cccc:c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$. We again see that $x = 1, w = 4$, and $z = 3y$.

(c) $2x + 3y + z - w = 1$
$x - y + w = 2$
$4x + y + z + w = 4$
$6x + 3y - 7z - w = 12$

The augmented matrix is $\left[\begin{array}{cccc:c} 2 & 3 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 2 \\ 4 & 1 & 1 & 1 & 4 \\ 6 & 3 & -7 & -1 & 12 \end{array}\right]$. The row operations $r_1 \leftrightarrow r_2, -2r_1 + r_2 \to r_2, -4r_1 + r_3 \to r_3, -6r_1 + r_4 \to r_4$ give us the matrix $\left[\begin{array}{cccc:c} 1 & -1 & 0 & 1 & 2 \\ 0 & 5 & 1 & -3 & -3 \\ 0 & 5 & 1 & -3 & -4 \\ 0 & 9 & -7 & -7 & 0 \end{array}\right]$. We can see looking at rows 2 and 3 that there will not be any solutions because we cannot have $5y + z - 3w$ equal to both -3 and -4. We can also see this by taking $r_2 - r_3 \to r_2$. Then r_2 would be $\left[\begin{array}{cccc:c} 0 & 0 & 0 & 0 & 1 \end{array}\right]$ and there are no solutions to the equation $0 = 1$.

9. Find the inverse of A or show that A is not invertible.

(a) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \\ 3 & 4 & 5 \end{bmatrix}$

Start with the partitioned matrix $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & 5 & 0 & 0 & 1 \end{array}\right]$. Perform the following row operations: $-2r_1+r_2 \to r_2$, $-3r_1+r_3 \to r_3$, $r_2 \leftrightarrow r_3$, $\frac{1}{4}r_2 \to r_2$, $-r_3 \to r_3$, $\frac{1}{4}r_3 + r_2 \to r_2$, and $-2r_3 + r_1 \to r_1$.

The resulting matrix is $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -1/4 & -1/4 & 1/4 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$, so the inverse of is $A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -1/4 & -1/4 & 1/4 \\ 2 & -1 & 0 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 7 & 5 \\ 3 & -1 & 2 \\ 5 & 13 & 12 \end{bmatrix}$

Start with the partitioned matrix $\left[\begin{array}{ccc|ccc} 1 & 7 & 5 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 5 & 13 & 12 & 0 & 0 & 1 \end{array}\right]$. If we do the row operations $-3r_1 + r_2 \to r_2, -5r_1 + r_3 \to r_3$, and $-r_2 + r_3 \to r_3$ we get $\left[\begin{array}{ccc|ccc} 1 & 7 & 5 & 1 & 0 & 0 \\ 0 & -22 & -13 & -3 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right]$. We can stop here because A is row equivalent to matrix with a row of zeros, so A is not invertible.

(c) $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

Start with the partitioned matrix $\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$. Perform the following row operations: $-2r_1 + r_4 \to r_4$, $r_2 \leftrightarrow r_3$, $-3r_2 + r_3 \to r_3$, $r_3 \leftrightarrow r_4$, $\frac{1}{2}r_3 \to r_3$, $-\frac{1}{12}r_4 \to r_4$, $-\frac{1}{2}r_4 + r_3 \to r_3$, $-4r_4 + r_2 \to r_2$, and $r_3 + r_1 \to r_1$.

The resulting matrix is $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1/24 & -1/8 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1/24 & -1/8 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & -1/12 & 1/4 & 0 \end{array}\right]$, so the inverse of is $A^{-1} = \frac{1}{24}\begin{bmatrix} 0 & 1 & -3 & 12 \\ 0 & 8 & 0 & 0 \\ -24 & 1 & -3 & 12 \\ 0 & -2 & 6 & 0 \end{bmatrix}$.

10. Let A be an $n \times n$ matrix such that the n-th row is a linear combination of rows 1 through $n-1$. Prove that A is not invertible.

    Row n is a linear combination of rows 1 through $n-1$ so there are constants $k_1, ..., k_{n-1}$ such that $r_n = k_1 r_1 + k_2 r_2 + ... + k_{n-1} r_{n-1}$ (where $r_1, r_2, ..., r_n$ are the rows of A).

    We can do type three row operations of the form $r_n - k_i r_i \to r_n$ for $i = 1, 2, ..., n-1$. The resulting matrix will have an n-th row which consists of all zeros. The determinant of A was not changed by the type three row operations and any matrix with a row of zeros has determinant 0, so $\det(A) = 0$ and A is not invertible.

11. For what value or values of k is the matrix $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & -2 & 1 \\ 2 & 6 & k \end{bmatrix}$ invertible?

    To check if A is invertible, we want to see if the RREF of A is I_3. If we do the row operations $r_1 \leftrightarrow r_2$, $r_2 - 2r_1 \to r_2, r_3 - 2r_1 \to r_3, r_3 - 2r_2 \to r_3$ we get $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & k-6 \end{bmatrix}$. We can see that if $k = 6$, the RREF of A will have a row of zeros and hence will not be I_3 and A will not be invertible. If $k \neq 6$, then we can get the REF by dividing row 2 by 5 and row 3 by $k-6$ and we can see already from the REF that there are leading 1's in each column so the RREF is going to be I_3 and A will be invertible. The values of k for which A is invertible are all $k \neq 6$.

    Another way to do this is with determinants. The determinant is $\det(A) = -4k + 2 + 24 - (-16) - 12 - k = -5k + 30$. A is invertible when this is nonzero, which is when $k \neq 6$.

12. Let A be a 4×4 matrix with $\det(A) = -4$. What is the RREF of A? How many solutions does the homogeneous system $A\mathbf{x} = \mathbf{0}$ have?

    As $\det(A) \neq 0$, A is invertible so it has RREF equal to I_4 and $A\mathbf{x} = \mathbf{0}$ has exactly one solution (the trivial solution). Note that it didn't matter what $\det(A)$ was, just that it was nonzero.

13. Suppose A is a 3×3 matrix with $\det(A) = 6$. Compute the determinant of the following matrices.

    (a) A^3
    $\det(A^3) = \det(A)^3 = 216$

    (b) $2A$
    $\det(2A) = 2^3 \det(A) = (8)(6) = 48$

(c) $(A^T)^{-1}$

$\det((A^T)^{-1}) = 1/\det(A^T) = 1/\det(A) = 1/6$.

14. Suppose A and B are invertible 3×3 matrices and $AB^T = 2B^2$. If $\det(A) = 5$, what is $\det(B)$?

$\det(AB^T) = \det(2B^2)$. The left side is $\det(AB^T) = \det(A)\det(B^T) = \det(A)\det(B) = 5\det(B)$. The right side is $\det(2B^2) = 2^3\det(B^2) = 8\det(B)\det(B) = 8\det(B)^2$. This gives us the equation $5\det(B) = 8\det(B)^2$. There are two possible solutions to this equation, $\det(B) = 5/8$ or $\det(B) = 0$, but B is invertible so $\det(B) \neq 0$ and thus $\det(B) = 5/8$.

15. Compute the determinant of A.

(a) $A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$

For 2×2 matrixes, the determinant is $a_{11}a_{22} - a_{12}a_{21}$ so $\det(A) = (3)(5) - (-1)(2) = 17$.

(b) $A = \begin{bmatrix} 0 & 1 & -2 \\ 5 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$

There are a lot of different methods that can be used to find this determinant. For review purposes we will go over all of them. On an exam, you can use whatever method you find easiest. Time permitting, you may want to use more than one method as a check.

Method 1: Using the definition of determinant. There are two ways to pick 3 nonzero entries so that we have exactly one from each row and column. We can take the 5 from column 1, the 1 from column 2, and the 3 from column 3 or we can take the 5 from column 1, the -1 from column 2, and the -2 from column 3. The first choice of three entries corresponds to the permutation 213 which has 1 inversion so is odd. The second choice corresponds to the permutation 312 which has 2 inversions so is even. The determinant is therefore $-(5)(1)(3) + (5)(-1)(-2) = -5$.

Method 2: Reduction to triangular form. The row operations $r_1 \leftrightarrow r_2$, $r_3 + r_2 \to r_3$ will give you the upper triangular matrix $\begin{bmatrix} 5 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. The determinant of an upper triangular matrix is the product of its diagonal entries so this matrix has determinant $(5)(1)(1) = 5$. The first row operation was type 1 so it swapped the sign of the determinant and the second was type 3 so it does not change the determinant so the determinant of the original matrix A is -5.

Method 3: Cofactor Expansion. Using cofactor expansion along the first column, we get that $\det(A) = -5\det\left(\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\right) = -5((1)(3) - (-2)(-1)) = -5$.

Method 4: We can also use the trick for 3×3 matrices where we repeat the first and second column next to A. See Example 8 in Section 3.1 for a more detailed explanation of this method.

(c) $A = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 3 & 4 & 7 \\ 0 & 0 & 5 & 8 \\ 2 & 0 & 0 & 9 \end{bmatrix}$

Method 1: Using the definition of determinant. We are looking for ways to pick out 4 nonzero entries so that we have exactly one from each row and each column. In column 2 we must take the second entry. In column 3, we cannot take the second entry since we already have something from row 2 so we must take the 3rd entry. Then we can either take the first entry from the first column and fourth entry from the fourth column, or we can take the fourth entry from column 1 and the first entry from column 4. We see that the determinant has 2 nonzero terms which are $a_{11}a_{22}a_{33}a_{44} = (1)(3)(5)(9) = 135$ and $a_{14}a_{22}a_{33}a_{41} = (6)(3)(5)(2) = 180$. We also need to determine the sign $\pm$ to go along with each term. The first term corresponds to the permutation 1234 which has no inversions so is even and gets a $+$. The second term corresponds to 4231 which has inversions $42, 43, 41, 21, 31$ so it has 5 inversions and is odd and gets a $-$. Hence $\det(A) = 135 - 180 = -45$.

Method 2: Reduction to triangular form. The single row operation $r_4 - 2r_1 \to r_4$ will result in the upper triangular matrix $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 3 & 4 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -3 \end{bmatrix}$. The determinant of an upper triangular matrix is the product of the diagonal entries so this matrix has determinant $(1)(3)(5)(-3) = -45$. As the row operation we did was a type 3 row operation, it does not change the determinant and hence $\det(A)$ is also -45.

Method 3: Cofactor expansion. Using cofactor expansion along the second column we get $\det(A) = 3\det\left(\begin{bmatrix} 1 & 0 & 6 \\ 0 & 5 & 8 \\ 2 & 0 & 9 \end{bmatrix}\right)$. Expanding along the first row, this is $3\left[1\det\left(\begin{bmatrix} 5 & 8 \\ 0 & 9 \end{bmatrix}\right) + 6\det\left(\begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}\right)\right] = 3(45 + -60) = 3(-15) = -45$.

Of course, you can also use a combination of methods. For example, you could use cofactor expansion to get down to 3×3 matrices then use the trick for 3×3 matrices to compute the 3×3 determinants.

16. The matrix $A = \begin{bmatrix} 1 & 2 & 6 & 8 \\ 1 & 3 & 0 & 9 \\ 1 & 4 & 0 & 10 \\ 1 & 5 & 7 & 0 \end{bmatrix}$ is invertible. Find all solutions to the following linear systems.

(a) $A^{-1}\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

If we multiply $A^{-1}\mathbf{x} = \mathbf{b}$ by A, we get $\mathbf{x} = A\mathbf{b}$ is the only solution. This is $A\mathbf{b} = \begin{bmatrix} -2 \\ 5 \\ 6 \\ 0 \end{bmatrix}$.

(b) $A\mathbf{x} = \mathbf{0}$

A is invertible so the only solution is the trivial solution, $\mathbf{x} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.