Bonus Quiz Solutions

1. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$. Find all eigenvalues of A. For each eigenvalue, find an associated eigenvector. (6 pts)

$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 2 & -1 \\ -3 & \lambda - 4 \end{bmatrix}\right) = (\lambda - 2)(\lambda - 4) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5)$. This is 0 when $\lambda = 1, 5$ so the eigenvalues of A are 1 and 5.

$\lambda = 1$: Here $\lambda I - A = \begin{bmatrix} -1 & -1 \\ -3 & -3 \end{bmatrix}$ which has RREF $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so the solutions to $(\lambda I - A)\mathbf{x} = \mathbf{0}$ are all vectors of the form $\begin{bmatrix} -b \\ b \end{bmatrix}$. The eigenvectors associated with $\lambda = 1$ are all vectors of the form $\begin{bmatrix} -b \\ b \end{bmatrix}$ with $b \neq 0$. For example $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\lambda = 5$: Here $\lambda I - A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$ so the RREF is $\begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$. The eigenvectors associated with $\lambda = 5$ are all vectors of the form $\begin{bmatrix} (1/3)b \\ b \end{bmatrix}$ with $b \neq 0$. For example, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

2. Suppose 3 is an eigenvalue of the matrix B and that $\mathbf{x}$ is an eigenvector associated with 3. Prove that $\mathbf{x}$ is also an eigenvector of $B + 2I$. What is the associated eigenvalue? (4 pts)

B has eigenvalue 3 with associated eigenvector $\mathbf{x}$ which means that $\mathbf{x} \neq \mathbf{0}$ and $B\mathbf{x} = 3\mathbf{x}$. Then $(B + 2I)\mathbf{x} = B\mathbf{x} + 2I\mathbf{x} = 3\mathbf{x} + 2\mathbf{x} = 5\mathbf{x}$. As $\mathbf{x} \neq \mathbf{0}$, this shows that $\mathbf{x}$ is an eigenvector of $B + 2I$ with associated eigenvalue 5.