Quiz 2 Solutions

1. Let V be the set of all ordered pairs of real numbers with the operations $(x, y) \oplus (x', y') = (x+x', y+y')$ and $r \odot (x, y) = (0, rxy)$. Check if this satisfies the following property of a vector space:
$(c + d) \odot \mathbf{u} = c \odot \mathbf{u} \oplus d \odot \mathbf{u}$ for any $\mathbf{u}$ in V and any real numbers c, d (3 pts)

   The elements of V look like (x, y) so replace $\mathbf{u}$ with (x, y) in the above property. The left side is $(c+d)\odot(x, y) = (0, (c+d)xy)$ and the right side is $c\odot(x, y)\oplus d\odot(x, y) = (0, cxy)\oplus(0, dxy) = (0, cxy+dxy) = (0, (c+d)xy)$. These are equal so the property is satisfied.

2. Let A be an $m \times n$ matrix and $\mathbf{b}$ be a nonzero m-vector. Is the set of all solutions to the linear system $A\mathbf{x} = \mathbf{b}$ a subspace of $\mathbb{R}^n$? Why or why not? (3 pts)

   This is not a subspace. The set of solutions to $A\mathbf{x} = \mathbf{b}$ is contained in $\mathbb{R}^n$, but is not closed under addition or scalar multiplication and could be the empty set. To show it's not closed under addition, let $\mathbf{x_1}$ and $\mathbf{x_2}$ be two solutions to $A\mathbf{x} = \mathbf{b}$. Then $A(\mathbf{x_1}+\mathbf{x_2}) = \mathbf{b}+\mathbf{b} = 2\mathbf{b} \neq \mathbf{b}$ as $\mathbf{b}$ is nonzero. To show it is not closed under scalar multiplication, if $\mathbf{x_1}$ is a solution and r is real number, then $A(r\mathbf{x_1}) = r(A\mathbf{x_1}) = r\mathbf{b}$.

   Note: You only need to show that it fails one of the two closure properties. Also, another argument that it is not a subspace would be that $\mathbf{0}$ is not in the set.

3. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$. Is S a spanning set for $\mathbb{R}^3$? (4 pts)

   This is not a spanning set. If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector in $\mathbb{R}^3$, then we want to check if there are constants x, y, z such that $x \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + z \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. This is the linear system with augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 1 & 1 & 2 & b \\ 2 & 3 & 1 & c \end{array}\right]$. The row operations $r_2 - r_1 \to r_2, r_3 - 2r_1 \to r_3, r_3 - r_2 \to r_3$ give you the matrix $\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & 3 & b-a \\ 0 & 0 & 0 & c-a-b \end{array}\right]$. We see that if $c - a - b \neq 0$ then there will be no solutions, so the span is not all of $\mathbb{R}^3$.

   In particular, the span is all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ with $c = a + b$.