Homework 9 Due: Friday, October 17

1. Find a basis for the subspace of P_3 spanned by

$$\{t^3+t, t-1, t^3+1, t^2+t, t^3+t^2+t+1\} \ .$$

Take a linear combination of the vectors and set it equal to zero: $a(t^3+t)+b(t-1)+c(t^3+1)+d(t^2+t)+e(t^3+t^2+t+1)=0$. This is equal to $(a+c+e)t^3+(d+e)t^2+(a+b+d+e)t+(-b+c+e)=0$ so we need $a+c+e=0, d+e=0, a+b+d+e=0, -b+c+e=0$. This is a homogeneous linear system with augmented matrix $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \end{array}\right]$. The row operations $r_3-r_1 \to r_3, r_4+r_3 \to r_4, r_4-r_2 \to r_4, r_2 \leftrightarrow r_3$ will get the following matrix in REF: $\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. The leading ones are in columns 1,2, and 4 so the first, second, and fourth vectors in the set form a basis. The basis we get this way is $\{t^3+t, t-1, t^2+t\}$.

2. Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 4 & 1 & 1 & 2 \\ -1 & 3 & -3 & 7 \end{bmatrix}$.

(a) Find a basis for the row space of A which consists of vectors which are row vectors of A.

Since we want the vectors to be row vectors, we can do this by forming the linear combination $a\begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}+b\begin{bmatrix}2 & 1 & -1 & 0\end{bmatrix}+c\begin{bmatrix}4 & 1 & 1 & 2\end{bmatrix}+d\begin{bmatrix}-1 & 3 & -3 & 7\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}$. This is the linear system $a+2b+4c-d=0, b+c+3d=0, a-b+c-3d=0, a+2c+7d=0$. This has $\left[\begin{array}{cccc|c} 1 & 2 & 4 & -1 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 \\ 1 & 0 & 2 & 7 & 0 \end{array}\right]$. The REF of this matrix is $\left[\begin{array}{cccc|c} 1 & 2 & 4 & -1 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. The leading ones are in columns 1,2, and 4 so we take basis

$$\left\{\begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}, \begin{bmatrix}2 & 1 & -1 & 0\end{bmatrix}, \begin{bmatrix}-1 & 3 & -3 & 7\end{bmatrix}\right\} \ .$$

(b) Find another basis for the row space of A which consists of vectors that are not row vectors of A.

Another way to find a basis for the row space of A is to take the nonzero rows of REF or RREF of A, as the row operations do not change the row space. The RREF of A is $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This gives us the basis

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix} \right\}.$$

(c) Is the vector $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ in the row space of A?

We can check this by seeing if $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ is a linear combination of the basis vectors for the row space. We can use either basis, but the basis from (b) will be easier to check. If $a \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 & 4 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ then $a = 1, b = 2, c = 3, -a + 4b + 2c = 4$. This has no solutions so $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ is not in the row space of A.

3. Let $A = \begin{bmatrix} 1 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & -1 & 3 \\ 2 & 2 & 7 & 3 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$.

(a) Find the rank and nullity of A.

We can find the rank and nullity by doing row operations to get to REF or RREF. The row operations $r_3 - 2r_1 \to r_3, -r_2 \to r_2, r_3 + 3r_4 \to r_3, r_4 \leftrightarrow r_2, r_3 \leftrightarrow r_4$ give the following matrix in REF $\begin{bmatrix} 1 & 1 & 5 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There are 3 leading ones, so the rank is 3. The nullity is the number of columns minus the rank so the nullity is 2.

(b) Find a basis for the row space of A.

One possible basis is the nonzero rows from REF which is

$$\left\{ \begin{bmatrix} 1 & 1 & 5 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & -3 \end{bmatrix} \right\}.$$

(c) Find a basis for the column space of A.

The leading ones of REF are in columns 1,3,4 so columns 1,3, and 4 of A

will be a basis for the column space. The basis we get is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}.$$

(d) Find a basis for the null space of A.

The row operations $r_2 + r_3 \to r_2, r_1 - 5r_2 \to r_1$ will get the REF matrix from (a) into RREF. The RREF of A is $\begin{bmatrix} 1 & 1 & 0 & 0 & 11 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Using variables $a, b, c, d, e,$ the solutions to $A\mathbf{x} = \mathbf{0}$ will be that b, e are anything and $d = 3e, c = 2e, a = -b - 11e$ so the solutions are all vectors of the form $\begin{bmatrix} -b - 11e \\ b \\ 2e \\ 3e \\ e \end{bmatrix} = b \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} -11 \\ 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}$. This has basis

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ 0 \\ 2 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

4. Let A be a 4×6 matrix with rank 4.

(a) Find the nullity of A.

The nullity is 2 (the number of columns minus the rank).

(b) How many solutions does $A\mathbf{x} = \mathbf{0}$ have?

The nullity is > 0 so there are nontrivial solutions and hence $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions.

(c) Find the dimension of the row space of A.

4

(d) Are the rows of A linearly independent?

Yes. The row space has dimension 4 and is spanned by the 4 rows of A so the rows of A must be a basis for the row space and are linearly independent.

(e) Find the dimension of the column space of A.

4

(f) Are the columns of A linearly independent?

No. There are 6 columns but their span only has dimension 4.

5. Let A be a 5×3 matrix with rank 3. Find the RREF of A.

The rank is 3 so there must be 3 leading ones in the RREF of A. As there are only 3 columns, the leading ones must be in the $(1,1)$, $(2,2)$, and $(3,3)$ positions. In RREF, the other entries in a column containing a leading one are all zero, so the RREF of A must be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

6. Let A be a 3×4 matrix. Write $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}$ for the rows of A. Suppose we know the following: $\mathbf{r_3} \neq \mathbf{0}$, $\mathbf{r_2}$ is not a multiple of $\mathbf{r_3}$, and $\mathbf{r_1}$ is contained in $\text{span}\{\mathbf{r_2}, \mathbf{r_3}\}$. Find the rank and nullity of A.

From the given information, we can find the dimension of the row space. The row space is $\text{span}\{\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}\}$. From the statement $\mathbf{r_1}$ is contained in $\text{span}\{\mathbf{r_2}, \mathbf{r_3}\}$, we know that the first row is a linear combination of the second and third, so $\text{span}\{\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}\} = \text{span}\{\mathbf{r_2}, \mathbf{r_3}\}$. The statements $\mathbf{r_3} \neq \mathbf{0}$ and $\mathbf{r_2}$ is not a multiple of $\mathbf{r_3}$ tell us that the set $\{\mathbf{r_2}, \mathbf{r_3}\}$ is linearly independent. It follows that $\{\mathbf{r_2}, \mathbf{r_3}\}$ is a basis for the row space of A, so the row space has dimension 2.

The rank of A is equal to the dimension of the row space so the rank is 2. The nullity is the number of columns minus the rank, so the nullity is also 2.

7. Let A be an $m \times n$ matrix with $m \neq n$. Show that either the rows of A are linearly dependent or the columns of A are linearly dependent.

If A has more rows than columns ($m > n$), then the rows must be linearly dependent since it is impossible to have m linearly independent vectors in $\mathbb{R}_n$ when $m > n$. If A has more columns than rows ($n > m$), then the columns must be linearly dependent because you cannot have n linearly vectors in $\mathbb{R}^m$ when $n > m$.