Homework 8 Due: Monday, October 13

1. Find a basis for and the dimension of the null space of $A = \begin{bmatrix} 2 & -2 & 1 & 7 \\ 1 & -1 & 1 & 4 \\ -3 & 3 & 8 & -1 \\ -2 & 2 & 8 & 2 \end{bmatrix}$.

   The null space of A is the set of all solutions to $A\mathbf{x} = \mathbf{0}$. The RREF of A is $\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The solutions are all vectors of the form $\begin{bmatrix} y-3w \\ y \\ -w \\ w \end{bmatrix}$ where y, w can be anything. This can be rewritten as $\begin{bmatrix} y-3w \\ y \\ -w \\ w \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ so the null space is spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$. These vectors are linearly independent, so this is a basis for the null space. The dimension of the null space is 2.

2. Let $A = \begin{bmatrix} 1 & -3 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

   (a) Find all real numbers λ such that $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

   $\lambda I - A = \begin{bmatrix} \lambda-1 & 3 & 0 \\ 1 & \lambda+1 & 0 \\ 0 & 0 & \lambda-2 \end{bmatrix}$. The homogeneous linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ will have a nontrivial solution if and only if $\det(\lambda I - A) = 0$. The determinant is $\det(\lambda I - A) = (\lambda-2)((\lambda-1)(\lambda+1)-3) = (\lambda-2)(\lambda^2-4) = (\lambda-2)^2(\lambda+2)$. This is 0 when $\lambda = \pm 2$.

   (b) For each λ from part (a), find a basis for the solution space of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

   If $\lambda = 2$, then $(\lambda I - A) = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which has RREF $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The null space of this matrix is all vectors of the form $\begin{bmatrix} -3y \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

which has basis $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

If $\lambda = -2$, then $(\lambda I - A) = \begin{bmatrix} -3 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ which has RREF $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The null space of this matrix is all vectors of the form $\begin{bmatrix} y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ which has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

3. Let A be a 3×3 matrix. Find all possible values for the nullity of A if:

   (a) $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

   The nullity is 1,2, or 3. It cannot be 0, as the solution space of $A\mathbf{x} = \mathbf{0}$ is not the zero vector space.

   (b) The RREF of A has 2 nonzero rows and one zero row.

   The nullity is 1. It is the same as the number of columns without leading ones. There are 2 nonzero rows so there are two leading ones and 3 columns and thus the nullity is 1.

   (c) A is row equivalent to a matrix B with nullity 2.

   The nullity is 2. If two matrices are row equivalent, then they have the same RREF so they have the same nullity.

   (d) A is not the zero matrix.

   The nullity is 0,1, or 2. The nullity cannot be 3 because this would mean the RREF of A had no leading ones so the RREF of A would be the zero matrix. However the zero matrix is only row equivalent to itself so if A is nonzero then the RREF of A is also nonzero.

   (e) $\det(A) \neq 0$
   The nullity is 0. If the determinant is nonzero then A is invertible so $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

4. The matrix $A = \begin{bmatrix} -1 & -2 & -5 & -11 & 5 \\ 3 & 4 & 13 & 29 & -4 \\ 2 & -2 & 4 & 10 & 6 \\ 1 & 1 & 4 & 9 & -2 \end{bmatrix}$ has RREF $\begin{bmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find all solutions to $A\mathbf{x} = \mathbf{0}$. Write your answer as a vector.

From the RREF, we get that the solutions are all vectors of the form $\begin{bmatrix} -3r-7s \\ -r-2s \\ r \\ s \\ 0 \end{bmatrix}$.

(b) Let $\mathbf{b} = \begin{bmatrix} 7 \\ -10 \\ 2 \\ -4 \end{bmatrix}$. Show that $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

Here we just need to multiply A by the vector to check that we get $\mathbf{b}$.

$$\begin{bmatrix} -1 & -2 & -5 & -11 & 5 \\ 3 & 4 & 13 & 29 & -4 \\ 2 & -2 & 4 & 10 & 6 \\ 1 & 1 & 4 & 9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \\ 2 \\ -4 \end{bmatrix}$$

(c) Use your answers to the previous 2 parts to find all solutions to $A\mathbf{x} = \mathbf{b}$. Write your answer as a vector.

The solutions are all vectors of the form $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3r-7s \\ -r-2s \\ r \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} 1-3r-7s \\ 1-r-2s \\ -1+r \\ s \\ 1 \end{bmatrix}$.

5. Let $S = \{t^2-1, t+2, 3t\}$ be an ordered basis for P_2.

(a) Find the coordinate vector of $p(t) = 3t^2 + t + 1$ with respect to S.

Find a, b, c such that $3t^2 + t + 1 = a(t^2-1) + b(t+2) + c(3t)$. This is equal to $a(t^2) + (b+3c)t + (-a+2b)$ so we need $3 = a, 1 = b + 3c, 1 = -a + 2b$. This has solution $a = 3, b = 2, c = -\frac{1}{3}$ so the coordinate vector is $\begin{bmatrix} 3 \\ 2 \\ -\frac{1}{3} \end{bmatrix}$.

(b) Find $q(t)$ where $q(t)$ has coordinate vector $\begin{bmatrix} 7 \\ -1 \\ 1 \end{bmatrix}$.

$q(t) = 7(t^2-1) - 1(t+2) + 1(3t) = 7t^2 + 2t - 9$

6. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$ be ordered bases for $\mathbb{R}^2$. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(a) Find the coordinate vectors $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$.

To find $[\mathbf{v}]_S$ we need to find a, b such that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so $1 = a - b, 0 = a + b$. This has solution $a = \frac{1}{2}$ and $b = -\frac{1}{2}$ so $[\mathbf{v}]_S = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

To find $[\mathbf{v}]_T$ we need to find a, b such that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ so $1 = -a, 0 = 2a + 3b$. This has solution $a = -1$ and $b = \frac{2}{3}$ so $[\mathbf{v}]_T = \begin{bmatrix} -1 \\ \frac{2}{3} \end{bmatrix}$.

(b) Find the transition matrix $P_{S \leftarrow T}$ from T to S.

To find this matrix, take the vectors in T and find their coordinate vectors with respect to S. The coordinate vector of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ with respect to S is $\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$ and the coordinate vector of $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ with respect to S is $\begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$. So $P_{S \leftarrow T} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}$.

(c) Find the transition matrix $Q_{T \leftarrow S}$ from S to T.

Take the vectors in S and find their coordinate vectors with respect to T. The coordinate vector of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with respect to T is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and the coordinate vector of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with respect to T is $\begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix}$ so $Q_{T \leftarrow S} = \begin{bmatrix} -1 & 1 \\ 1 & -\frac{1}{3} \end{bmatrix}$.

(d) Verify that $[\mathbf{v}]_S = P_{S \leftarrow T}[\mathbf{v}]_T$ and $[\mathbf{v}]_T = Q_{T \leftarrow S}[\mathbf{v}]_S$.

$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} -1 \\ \frac{2}{3} \end{bmatrix}$ and $\begin{bmatrix} -1 \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

(e) How are $P_{S \leftarrow T}$ and $Q_{T \leftarrow S}$ related?

They are inverses.

7. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ be an ordered basis for $\mathbb{R}^3$. Suppose T is an

ordered basis for $\mathbb{R}^3$ such that $P_{S\leftarrow T} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 2 \\ 1 & 0 & 2 \end{bmatrix}$ and $\mathbf{v}$ is a vector in $\mathbb{R}^3$ with $[\mathbf{v}]_T = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

(a) Find the coordinate vector $[\mathbf{v}]_S$.

$$[\mathbf{v}]_S = P_{S\leftarrow T}[\mathbf{v}]_T = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 2 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 9 \end{bmatrix}.$$

(b) Find the basis T.

If $T = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, then column 1 of $P_{S\leftarrow T}$ is $[\mathbf{v_1}]_S$ so $\mathbf{v_1} = 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$. Similarly, $\mathbf{v_2} = 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v_3} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$. So $T = \left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} \right\}$.

(c) Find $\mathbf{v}$.

We can find $\mathbf{v}$ using $[\mathbf{v}]_T$ and T as follows: $\mathbf{v} = 1\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 29 \\ 20 \\ 8 \end{bmatrix}$.

We could also find it using $[\mathbf{v}]_S$ and S as follows: $\mathbf{v} = 8\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 12\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 29 \\ 20 \\ 8 \end{bmatrix}$.