1. S is a set of vectors in the vector space V. Determine if S is linearly independent. If it is not linearly independent, write one of the vectors in S as a linear combination of the other vectors in S.

   (a) $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

   If $x \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + w \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, this is the linear system $x + y + 3z + 2w = 0, 2x + y + 2z + w = 0, x + y + z = 0$. This has augmented matrix $\left[\begin{array}{cccc|c} 1 & 1 & 3 & 2 & 0 \\ 2 & 1 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 & 0 \end{array}\right]$ which has RREF $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$. There are infinite solutions to this linear system so the vectors are not linearly independent. In particular, the solutions are all vectors of the form $\begin{bmatrix} z \\ -4z \\ z \\ 0 \end{bmatrix}$.

   If we take $z = 1$, we see that $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ which gives us the third vector as a linear combination of the other vectors.

   (b) $V = M_{22}$, $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right\}$

   If $x \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + z \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then $x + 2y + 3z = 0, x = 0, 2y = 0, 3z = 0$. The only solution to this system is $x = y = z = 0$ so the set is linearly independent.

   (c) $V = \mathbb{R}^4$, $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 4 \end{bmatrix} \right\}$

   If $x \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ then $x + z = 0, 2x + 2w = 0, 3y + 3z = 0, 3y + 4w = 0$. The first two equations tell us that $z = -x, w = -x$

and by the third equation $y = -z = x$. Plugging these into the last equation gives $x = 0$ which forces $y = z = w = 0$. The only solution is $x = y = z = w = 0$ so the set is linearly independent.

(d) $V = P_2$, $S = \{t^2 - 1, t - 1, t + 1\}$

If $x(t^2 - 1) + y(t - 1) + z(t + 1) = 0$ then $xt^2 + (y + z)t + (-x - y + z) = 0$ which only happens if $x = 0, y + z = 0, -x - y + z = 0$. The only solution to this is $x = y = z = 0$ so the set is linearly independent.

2. Suppose that $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is a linearly independent set of vectors in a vector space V. Determine if each of the following sets is linearly independent.

(a) $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$

This set is linearly independent. Any subset of a linearly independent set is linearly independent.

(b) $\{\mathbf{v_1}, \mathbf{v_3}, \mathbf{v_3} + 2\mathbf{v_1}, \mathbf{v_4}\}$

This set is not linearly independent since one of the vectors is a linear combination of some of the other vectors.

(c) $\{\mathbf{v_1}, \mathbf{v_1} + \mathbf{v_2}, \mathbf{v_3} - \mathbf{v_4}\}$

If $x\mathbf{v_1} + y(\mathbf{v_1} + \mathbf{v_2}) + z(\mathbf{v_3} - \mathbf{v_4}) = \mathbf{0}$ then $(x + y)\mathbf{v_1} + y\mathbf{v_2} + z\mathbf{v_3} - z\mathbf{v_4} = \mathbf{0}$. The vectors in S are linearly independent so for this to be equal to the zero vector, we would need $x + y = 0, y = 0, z = 0, -z = 0$ and this system has only the solution $x = y = z = 0$ so vectors are linearly independent.

3. Let $\mathbf{v_1} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$.

(a) The equation $x\mathbf{v_1} + y\mathbf{v_2} + z\mathbf{v_3} = \mathbf{0}$ can be rewritten as a homogeneous linear system in 3 equations and 3 unknowns. Find the coefficient matrix A of this linear system. How does A relate to $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$?

This is the linear system $3x + y - z = 0, 2x + y + 2z = 0, x + 6z = 0$ which has coefficient matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 1 & 2 \\ 1 & 0 & 6 \end{bmatrix}$. The columns of A are exactly the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

(b) Compute $\det(A)$. What does $\det(A)$ tell us about the number of solutions to the homogeneous linear system?

$\det(A) = 9$. This is nonzero, so A is invertible and the homogeneous linear system has only the trivial solution.

(c) Use the previous parts to determine if $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly independent set.

The previous parts showed that the only solution to $x\mathbf{v_1} + y\mathbf{v_2} + z\mathbf{v_3} = \mathbf{0}$ is $x = y = z = 0$ so the set is linearly independent.

4. For each S and V from Problem 1, determine if S is a basis for V.

(a) S is not linearly independent so it is not a basis for V.

(b) M_{22} is a 4 dimensional space and S contains only 3 vectors so S is too small to span M_{22} and thus S is not a basis for M_{22}. In particular, $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ is an example of a matrix which is in M_{22} but is not in the span of S.

(c) This is a basis for $\mathbb{R}^4$ because the dimension of $\mathbb{R}^4$ is 4 and S is a linearly independent set of size 4.

(d) This is a basis for P_2 because the dimension of P_3 is 3 and S is a linearly independent set of size 3.

5. Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \\ 3 \end{bmatrix} \right\}$. Find a basis for the span of S. What is the dimension of span S?

The linear combination $a_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + a_5 \begin{bmatrix} 3 \\ 6 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ gives us the homogeneous linear system with augmented matrix $\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 0 & 3 & 0 \\ 2 & 0 & 2 & 1 & 6 & 0 \\ 0 & 1 & 1 & 1 & 3 & 0 \\ 0 & 3 & 3 & 0 & 3 & 0 \end{array}\right]$.

The REF of this matrix is $\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. The columns with leading ones are 1,2, and 4 so we can take the first, second, and fourth vectors as our basis. A basis for the span is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and the dimension is 3.

6. Find a basis for the following spaces. What are the dimensions of these spaces?

   (a) All vectors in $\mathbb{R}^3$ of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $a + c = 2b$.

   We can rewrite these vectors as vectors of the form $\begin{bmatrix} 2b - c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ so the space is spanned by $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. The two vectors $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ are linearly independent so they are a basis for the space. The basis has size 2 so it is dimension 2.

   Note that if you instead solved for b you would have gotten $\left\{ \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$.

   Or if you solved for c you would get $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$. These are also correct bases for this space.

   (b) All 3×3 skew symmetric matrices.

   The 3×3 skew symmetric matrices look like $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$, where a, b, c are anything. We can rewrite this as $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. It follows that $\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$ spans the space. These vectors are also linearly independent, so this is a basis for the space. The dimension is 3.

(c) All vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} a+b \\ a-c \\ b+c \\ -a-b \end{bmatrix}$.

Rewrite this as $a\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$. The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a spanning set for the space but it is not linearly independent however as the last vector is equal to the second one minus the first. We can delete the third vector without changing the span and the resulting set is linearly independent so a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$. The dimension is 2.

Note that in this case, any one of the vectors can be written as linear combination of the other two, so we could have deleted any one of the three vectors to get a basis.

7. Prove that if W is a subspace of a finite dimensional vector space V then $\dim W \leq \dim V$.

   Hint: The following theorem from Section 4.6 may be useful.

   Theorem 4.11: If S is a linearly independent set of vectors in a finite dimensional vector space V, then there is a basis T for V that contains S.

   Let S be a basis for W. As W is contained in V, the vectors in S are also in V. Also, as S is a basis, it is linearly independent. By the above theorem, S is contained in in a basis T for V. As S is contained in T, the size of S is less than or equal to the size of T. The dimension of a space is the size of a basis so $\dim W \leq \dim V$.

8. What are the possible dimensions for subspaces of $\mathbb{R}^3$? Describe what the subspaces of $\mathbb{R}^3$ of each dimension look like.

   The possible dimensions are $0, 1, 2, 3$. The only dimension 0 subspace is the zero vector space $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ which is just the origin. The 1 dimensional spaces are the lines through the origin. The 2 dimensional spaces are planes through the origin. The only 3 dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.