Homework 7 Due: Wednesday, October 8

1. S is a set of vectors in the vector space V. Determine if S is linearly independent. If it is not linearly independent, write one of the vectors in S as a linear combination of the other vectors in S.

   (a) $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

   (b) $V = M_{22}$, $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right\}$

   (c) $V = \mathbb{R}^4$, $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 4 \end{bmatrix} \right\}$

   (d) $V = P_2$, $S = \{t^2 - 1, t - 1, t + 1\}$

2. Suppose that $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is a linearly independent set of vectors in a vector space V. Determine if each of the following sets is linearly independent.

   (a) $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$

   (b) $\{\mathbf{v_1}, \mathbf{v_3}, \mathbf{v_3} + 2\mathbf{v_1}, \mathbf{v_4}\}$

   (c) $\{\mathbf{v_1}, \mathbf{v_1} + \mathbf{v_2}, \mathbf{v_3} - \mathbf{v_4}\}$

3. Let $\mathbf{v_1} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$.

   (a) The equation $x\mathbf{v_1} + y\mathbf{v_2} + z\mathbf{v_3} = \mathbf{0}$ can be rewritten as a homogeneous linear system in 3 equations and 3 unknowns. Find the coefficient matrix A of this linear system. How does A relate to $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$?

   (b) Compute $\det(A)$. What does $\det(A)$ tell us about the number of solutions to the homogeneous linear system?

   (c) Use the previous parts to determine if $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly independent set.

4. For each S and V from Problem 1, determine if S is a basis for V.

5. Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \\ 3 \end{bmatrix} \right\}$. Find a basis for the span of S. What is the dimension of span S?

6. Find a basis for the following spaces. What are the dimensions of these spaces?

   (a) All vectors in $\mathbb{R}^3$ of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $a + c = 2b$.

   (b) All 3×3 skew symmetric matrices.

   (c) All vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} a+b \\ a-c \\ b+c \\ -a-b \end{bmatrix}$.

7. Prove that if W is a subspace of a finite dimensional vector space V then $\dim W \leq \dim V$.

   Hint: The following theorem from Section 4.6 may be useful.

   Theorem 4.11: If S is a linearly independent set of vectors in a finite dimensional vector space V, then there is a basis T for V that contains S.

8. What are the possible dimensions for subspaces of $\mathbb{R}^3$? Describe what the subspaces of $\mathbb{R}^3$ of each dimension look like.