Homework 4 Due: Monday, September 15

Solutions to Additional Problem:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ -2 & 0 & 2 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Compute $\det(A)$ three different ways:

(1) using the definition of determinant

There are 4 ways to pick out exactly one nonzero entry from each row an column, so there will be four nonzero terms in the sum. The four ways to do this look as follows:

$$\begin{bmatrix} \textcircled{1} & 2 & 0 & 1 \\ 0 & \textcircled{1} & 2 & 0 \\ -2 & 0 & \textcircled{2} & -3 \\ 0 & 0 & 1 & \textcircled{3} \end{bmatrix}, \begin{bmatrix} \textcircled{1} & 2 & 0 & 1 \\ 0 & \textcircled{1} & 2 & 0 \\ -2 & 0 & 2 & \textcircled{-3} \\ 0 & 0 & \textcircled{1} & 3 \end{bmatrix}, \begin{bmatrix} 1 & \textcircled{2} & 0 & 1 \\ 0 & 1 & \textcircled{2} & 0 \\ \textcircled{-2} & 0 & 2 & -3 \\ 0 & 0 & 1 & \textcircled{3} \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & \textcircled{1} \\ 0 & \textcircled{1} & 2 & 0 \\ \textcircled{-2} & 0 & 2 & -3 \\ 0 & 0 & \textcircled{1} & 3 \end{bmatrix}.$$

The first one is the term $a_{11}a_{22}a_{33}a_{44} = (1)(1)(2)(3) = 6$. The corresponding permutation can be gotten by going through the rows in order and taking the column that has the circled entry. So for this one, the permutation is 1234. This has no inversions so it is even and this term gets a $(+)$. The second one is $a_{11}a_{22}a_{34}a_{43} = (1)(1)(-3)(1) = -3$. This has permutation 1243 which has one inversion (43) so it is an odd permutation and this term gets a $(-)$. The third term is $a_{12}a_{23}a_{31}a_{44} = (2)(2)(-2)(3) = -24$. This has permutation 2314 which has 2 inversions $(21, 31)$ so it is an even permutation and gets a $(+)$. The fourth term is $a_{14}a_{22}a_{31}a_{43} = (1)(1)(-2)(1) = -2$. This has permutation 4213 which has 4 inversions $(42, 41, 43, 21)$ and is even and gets a $(+)$. Putting these together, $\det(A) = (+)(6) + (-)(-3) + (+)(-24) + (+)(-2) = -17$.

(2) using reduction to triangular form

The following row operations will take A to an upper triangular matrix: $r_3 + 2r_1 \to r_3$, $r_3 - 4r_2 \to r_3$, $r_3 \leftrightarrow r_4$, $r_4 + 6r_3 \to r_4$. The resulting matrix is $B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 17 \end{bmatrix}$.

The determinant of an upper triangular matrix is the product of the diagonal entries so $\det(B) = 17$. The first two row operations were type 3 and did not change the determinant, the third was type 1 and changed the sign of the determinant, and the last was type 3 and did not change the determinant so $\det(B) = -\det(A)$ and hence $\det(A) = -17$.

(3) using cofactor expansion

You can expand along any row or column, although it is usually a good idea to pick a row or column with a lot of zeros. If we expand along row 2, we get that $\det(A) = (+)(1)\det(M_{22}) + (-)(2)\det(M_{23})$ where M_{22} is the matrix gotten from A by deleting row 2 and column 2 and M_{23} is the matrix gotten by deleting row 2 and column 3. These matrices are $M_{22} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 2 & -3 \\ 0 & 1 & 3 \end{bmatrix}$ and $M_{23} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & -3 \\ 0 & 0 & 3 \end{bmatrix}$. These determinants can also be computed using cofactor expansion. For the first one, if we expand along row 1 we get $\det(M_{22}) = (+)(1)\det\left(\begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix}\right) + (+)(1)\det\left(\begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix}\right) = 1(6-(-3)) + (1)(-2) = 7$. For the second matrix, expand along row 3 to get $\det(M_{23}) = (+)(3)\det\left(\begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}\right) = 3(0-(-4) = 12$. Plugging these into $\det(A) = (+)(1)\det(M_{22}) + (-)(2)\det(M_{23})$ we get $\det(A) = 1(7) - 2(12) = -17$.