Solutions to Additional Problems:

1. (a) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$. Compute AB.

   $$AB = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 21 \end{bmatrix}$$

   (b) Let C be an $n \times n$ diagonal matrix with diagonal entries $c_1, c_2, ..., c_n$ and D be an $n \times n$ diagonal matrix with diagonal entries $d_1, d_2, ..., d_n$. Describe the matrix CD.

   CD is also $n \times n$ and diagonal. The diagonal entries are $c_1d_1, c_2d_2, ..., c_nd_n$.

   (c) Determine if the following statement is true or false. Either explain why it is always true, or present a counterexample to show it is false.
   If C and D are diagonal $n \times n$ matrices then $CD = DC$.

   True. Use part (b). Both CD and DC are diagonal matrices. The entries on the diagonal of CD are $c_1d_1, c_2d_2, ..., c_nd_n$ and the entries on the diagonal of DC are $d_1c_1, d_2c_2, ..., d_nc_n$. The c_i and d_i are just numbers so $c_id_i = d_ic_i$ for all i and the entries of CD and DC are all equal.

2. (a) Write down an example of a 3×3 upper triangular matrix which is not diagonal and a 3×3 lower triangular matrix which is not diagonal.

   Lots of possible answers. Here's one possible answer, upper triangular and not diagonal: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, lower triangular and not diagonal: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

   (b) Determine if the following statement is true or false. Either explain why it is always true, or present a counterexample to show it is false.
   If A is an upper triangular $n \times n$ matrix and B is a lower triangular $n \times n$ matrix then AB is a diagonal matrix.

   False. There is a high likelihood that your example from (a) is a counterexample. If we take A, B as in the solution to (a), the matrix AB is not diagonal because the $1, 2$ entry is 1, so this is a counterexample.

3. Determine if each matrix is symmetric, skew symmetric, both, or neither.

   (a) AA^T where A is an $n \times m$ matrix

   Take the transpose of AA^T and see if the result is AA^T (symmetric), $-AA^T$ (skew symmetric), or neither. Here, $(AA^T)^T = (A^T)^TA^T = AA^T$ so it is symmetric.

(b) $A + A^T$ where A is an $n \times n$ matrix
$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$ so this is symmetric.

(c) $A - A^T$ where A is an $n \times n$ matrix
$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$ so this is skew symmetric.

(d) AB where A and B are $n \times n$ symmetric matrices
$(AB)^T = B^T A^T = BA$. This is neither since BA is not necessarily equal to AB or $-AB$.

(e) A^3 where A is an $n \times n$ skew symmetric matrix
$(A^3)^T = (A^T)^3 = (-A)^3 = (-1)^3 A = -A$ so this is skew symmetric.

4. Suppose A and B are $n \times n$ matrices such that $AB = 0$. Prove the following statements about A and B.

(a) If A is invertible then $B = 0$.

Assume that A is invertible and show that this forces $B = 0$. A is invertible so there exists A^{-1} with $AA^{-1} = A^{-1}A = I$. Starting with the equation $AB = 0$, multiply both sides by A^{-1} on the left to get $A^{-1}AB = A^{-1}0$. The left hand side simplifies to $A^{-1}AB = IB = B$. Anything times the zero matrix is the zero matrix so the right hand side is 0 and the equation becomes $B = 0$.

(b) If $B \neq 0$ then A is not invertible.

This is the contrapositive of part (a) so it is logically equivalent to (a). We proved that (a) was true so (b) is true as well.