1. Let $L : P_1 \to P_1$ be the linear operator $L(at+b) = (5a-4b)t + (2a-b)$. Prove that L is diagonalizable. Find a basis S for P_1 such that the representation of L with respect to S is diagonal.

The linear operator L will be diagonalizable if and only if a representation of L is a diagonalizable matrix. Let $T = \{t, 1\}$ be the standard basis for P_1 and let A be the representation of L with respect to T. We will show that A is a diagonalizable matrix. $L(t) = 5t+2$ and $L(1) = -4t-1$. The coordinates of these vectors with respect to T are $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$ so $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$. Then $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-5 & 4 \\ -2 & \lambda+1 \end{bmatrix}\right) = (\lambda-5)(\lambda+1)+8 = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$ so the eigenvalues of A (and of L) are 1 and 3. A is a 2×2 matrix with 2 distinct eigenvalues so it must be diagonalizable.

To find S, we will find the eigenvectors of A and use them to find the eigenvectors of L. For the eigenvalue $\lambda = 1$, the eigenspace is the solutions to the homogeneous linear system with coefficient matrix $\begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix}$. The RREF of this matrix is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so the solutions are all vectors of the form $\begin{bmatrix} b \\ b \end{bmatrix}$. The eigenspace of L associated with $\lambda = 1$ is therefore all vectors of the form $bt + b$. For $\lambda = 3$, the eigenspace is the solutions to the homogeneous linear system with coefficient matrix $\begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$. The RREF of this matrix is $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ so the eigenspace is all vectors of the form $\begin{bmatrix} 2b \\ b \end{bmatrix}$. The eigenspace of L associated with $\lambda = 3$ is all vectors of the form $2bt + b$. To get S, we take a basis for each eigenspace and put them together to get a basis for P_2. The resulting basis is $S = \{t+1, 2t+1\}$.

Check: We can check that this works by finding the representation of L with respect to S and seeing if it is diagonal. $L(t+1) = t+1$ which has coordinate $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with respect to S. $L(2t+1) = 6t+3 = 3(2t+1)$ which has coordinate vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ with respect to S. The representation with respect to S is therefore $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ which is diagonal (and the diagonal entries are the eigenvalues).

2. Determine if each of the following matrices is diagonalizable. Explain why or why not.

(a) $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

This matrix is diagonalizable. $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 & 1 \\ -1 & \lambda & -1 \\ -1 & 1 & \lambda - 2 \end{bmatrix}\right) =$ $\lambda^2(\lambda-2) - 1 - 1 + \lambda + \lambda - (\lambda - 2) = \lambda^2(\lambda-2) + \lambda = \lambda(\lambda(\lambda-2)+1) =$ $\lambda(\lambda^2 - 2\lambda + 1) = \lambda(\lambda-1)^2$. The eigenvalues are 0 with multiplicity 1 and 1 with multiplicity 2. We need to check that the dimension of the associated eigenspaces matches the multiplicities. The eigenspaces always have dimension at least 1 and never had dimension more than the multiplicity, so the dimension will match the multiplicity for any multiplicity 1 eigenvalues. We therefore just need to check the dimension of the eigenspace associated with $\lambda = 1$. The eigenspace associated with $\lambda = 1$ is the solutions to the homogeneous system with coefficient matrix $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$.
This has RREF $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. There are two columns without leading ones so the dimension of the eigenspace will be 2, which matches the multiplicity of $\lambda = 1$. The matrix is therefore diagonalizable.

(b) $B = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$

This matrix is not diagonalizable. $\det(\lambda I - B) = \det\left(\begin{bmatrix} \lambda & 0 & 0 & 1 \\ 0 & \lambda & 1 & -1 \\ 1 & -1 & \lambda & 0 \\ 1 & -1 & 0 & \lambda \end{bmatrix}\right)$.
Using cofactor expansion, this is equal to $\lambda \det\left(\begin{bmatrix} \lambda & 1 & -1 \\ -1 & \lambda & 0 \\ -1 & 0 & \lambda \end{bmatrix}\right) - \det\left(\begin{bmatrix} 0 & \lambda & 1 \\ 1 & -1 & \lambda \\ 1 & -1 & 0 \end{bmatrix}\right) =$ $\lambda(\lambda^3 - \lambda + \lambda) - (\lambda^2 - 1 + 1) = \lambda^4 - \lambda^2 = \lambda^2(\lambda^2 - 1) = \lambda^2(\lambda+1)(\lambda-1)$. The eigenvalues are 0 with multiplicity 2 and ± 1 each with multiplicity 1. We need to check if the dimension of the eigenspace associated with $\lambda = 0$ is 2. The eigenspace associated with $\lambda = 0$ is the solutions to the

homogenous linear system with coefficient matrix $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$. This has RREF $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. There is only one column without a leading 1 so the dimension of the eigenspace will be 1. This is smaller than the multiplicity so the matrix is not diagonalizable.

(c) $C = \begin{bmatrix} 1 & 2 & -4 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

This matrix is diagonalizable. $\det(\lambda I - C) = \det\left(\begin{bmatrix} \lambda-1 & -2 & 4 & 0 \\ 0 & \lambda+1 & -1 & -3 \\ 0 & 0 & \lambda-5 & -2 \\ 0 & 0 & 0 & \lambda+3 \end{bmatrix}\right) =$ $(\lambda-1)(\lambda+1)(\lambda-5)(\lambda+3)$. The eigenvalues are $1, -1, 5, -3$. C is a 4×4 matrix with 4 distinct eigenvalues, so it is diagonalizable.

3. Let A be a 3×3 matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = 4$. Suppose that $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ are eigenvectors of A such that $\mathbf{v_i}$ is associated with λ_i.

(a) Find a diagonal matrix D such that A is similar to D.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Note: The order of the diagonal entries doesn't matter, so technically there are 6 possible correct answers to this problem.

(b) Find an invertible matrix P such that $D = P^{-1}AP$.

The matrix P will have columns equal to the eigenvectors, so $P = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 3 & -5 & 2 \end{bmatrix}$.

Note: The order of the columns of P must match the ordering that you chose for the eigenvalues in D.

(c) Find a formula for A^k for k a positive integer.

$D = P^{-1}AP$ so $A = PDP^{-1}$. Then $A^k = (PDP^{-1})^k = PD^kP^{-1}$. Note that $D^k = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & 0^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4^k \end{bmatrix}$. To compute $P^{-1}D^kP$, we need to find P^{-1}. Start with the matrix $[P : I]$ and do row operations to get to $[I : P^{-1}]$. The inverse is $P^{-1} = \begin{bmatrix} 7 & 4 & -2 \\ 3 & 2 & -1 \\ -3 & -1 & 1 \end{bmatrix}$. Therefore $A^k =$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 3 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4^k \end{bmatrix} \begin{bmatrix} 7 & 4 & -2 \\ 3 & 2 & -1 \\ -3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 4^k \\ 3 & 0 & 2(4^k) \end{bmatrix} \begin{bmatrix} 7 & 4 & -2 \\ 3 & 2 & -1 \\ -3 & -1 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} 7 & 4 & -2 \\ -3(4^k) & -(4^k) & 4^k \\ 21 - 6(4^k) & 12 - 2(4^k) & -6 + 2(4^k) \end{bmatrix}.$$

(d) Find A and A^{50}.

From the previous part, we can find A by plugging in $k = 1$. So $A = \begin{bmatrix} 7 & 4 & -2 \\ -12 & -4 & 4 \\ -3 & 4 & 2 \end{bmatrix}$. To find A^{50}, plug in $k = 50$. So

$$A^{50} = \begin{bmatrix} 7 & 4 & -2 \\ -3(4^{50}) & -(4^{50}) & 4^{50} \\ 21 - 6(4^{50}) & 12 - 2(4^{50}) & -6 + 2(4^{50}) \end{bmatrix}.$$

4. Suppose A is a diagonalizable matrix. Prove that the following matrices are also diagonalizable.

Note: A is diagonalizable so it is similar to a diagonal matrix D. We can therefore write $A = B^{-1}DB$ where B is invertible. We will use this in all four parts.

(a) rA for any real number r

$A = B^{-1}DB$ so $rA = rB^{-1}DB = B^{-1}(rD)B$. All scalar multiples of diagonal matrices are diagonal so rD is also diagonal and this shows that

rA is similar to a diagonal matrix rD.

(b) A^k for any positive integer k

$A = B^{-1}DB$ so $A^k = (B^{-1}DB)^k = B^{-1}D^kB$. All powers of a diagonal matrix are also diagonal, so A^k is similar to the diagonal matrix D^k.

(c) A^T

$A = B^{-1}DB$ so $A^T = (B^{-1}DB)^T = B^TD^T(B^{-1})^T = B^TD(B^T)^{-1}$. Note that D is symmetric so $D^T = D$. This proves that A^T is also similar to D.

(d) A^{-1} (if A is invertible)

$A = B^{-1}DB$. Suppose A is invertible. Then D must be invertible as well (since similar matrices have the same determinant). The diagonal entries of D must all be nonzero and D^{-1} is the diagonal matrix whose entries are the reciprocals of the diagonal entries of D. Then $A^{-1} = (B^{-1}DB)^{-1} = B^{-1}D^{-1}(B^{-1})^{-1} = B^{-1}D^{-1}B$. This shows that A^{-1} is similar to the diagonal matrix D^{-1}.