1. Let $L : P_1 \to P_1$ be the linear operator $L(at + b) = (5a - 4b)t + (2a - b)$. Prove that $L$ is diagonalizable. Find a basis $S$ for $P_1$ such that the representation of $L$ with respect to $S$ is diagonal.

2. Determine if each of the following matrices is diagonalizable. Explain why or why not.

   (a) $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$

   (c) $C = \begin{bmatrix} 1 & 2 & -4 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

3. Let $A$ be a $3 \times 3$ matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = 4$. Suppose that $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ are eigenvectors of $A$ such that $\mathbf{v_i}$ is associated with $\lambda_i$.

   (a) Find a diagonal matrix $D$ such that $A$ is similar to $D$.

   (b) Find an invertible matrix $P$ such that $D = P^{-1}AP$.

   (c) Find a formula for $A^k$ for $k$ a positive integer.

   (d) Find $A$ and $A^{50}$.

4. Suppose $A$ is a diagonalizable matrix. Prove that the following matrices are also diagonalizable.

   (a) $rA$ for any real number $r$

   (b) $A^k$ for any positive integer $k$

   (c) $A^T$

   (d) $A^{-1}$ (if $A$ is invertible)