1. Let $L : P_1 \to P_1$ be the linear operator $L(at + b) = (b - a)t + 5b$. Find all eigenvalues of L and all associated eigenvectors.

   If $L(at+b) = \lambda(at+b)$ then $(b-a)t+5b = \lambda(at+b)$ so we get the two equations $b - a = \lambda a$ and $5b = \lambda b$. From the second equation, either $\lambda = 5$ or $b = 0$. If $\lambda = 5$ then $b - a = 5a$ so $b = 6a$. If $b = 0$ then the first equation becomes $-a = \lambda a$ so $a = 0$ or $\lambda = -1$. We cannot have both a and b equal to 0 since $\mathbf{0}$ is not an eigenvector, so in the $b = 0$ case we must have $\lambda = -1$ and a can be anything.

   The eigenvalues of L are $\lambda = 5$ and $\lambda = -1$. The eigenvectors associated with $\lambda = 5$ are all vectors of the form $at+6a$ with $a \neq 0$. The eigenvectors associated with $\lambda = -1$ are all vectors of the form at with $a \neq 0$.

2. Let $A = \begin{bmatrix} 1 & 5 & 2 & -5 & c \\ 1 & 2 & 3 & 4 & 0 \\ 3 & -6 & 11 & 1 & 1 \\ 2 & 2 & 2 & 1 & 3 \\ 3 & 9 & 6 & -8 & 0 \end{bmatrix}$. For what value or values of c is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ an eigenvector of A? What is the associated eigenvalue?

   $$A\mathbf{x} = \begin{bmatrix} 1 & 5 & 2 & -5 & c \\ 1 & 2 & 3 & 4 & 0 \\ 3 & -6 & 11 & 1 & 1 \\ 2 & 2 & 2 & 1 & 3 \\ 3 & 9 & 6 & -8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+c \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}.$$ This is a multiple of $\mathbf{x}$ when $c = 7$, in which case it is $10\mathbf{x}$. Therefore $\mathbf{x}$ is an eigenvector when $c = 7$ and the associated eigenvalue is 10.

3. Find the eigenvalues of A. For each eigenvalue, find a basis for the associated eigenspace.

   (a) $A = \begin{bmatrix} 0 & -3 & -1 \\ -1 & 2 & 1 \\ 3 & -9 & -4 \end{bmatrix}$

$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 3 & 1 \\ 1 & \lambda-2 & -1 \\ -3 & 9 & \lambda+4 \end{bmatrix}\right) = \lambda(\lambda-2)(\lambda+4) + 9\lambda = \lambda((\lambda-2)(\lambda-4)+9) = \lambda(\lambda^2+2\lambda+1) = \lambda(\lambda+1)^2$. The eigenvalues are -1 and 0.

For $\lambda = -1$, the associated eigenspace is solutions to $(-I - A)\mathbf{x} = \mathbf{0}$ which has augmented matrix $\left[\begin{array}{ccc|c} -1 & 3 & 1 & 0 \\ 1 & -3 & -1 & 0 \\ -3 & 9 & 3 & 0 \end{array}\right]$. The RREF of this matrix is $\left[\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, so if the vector is $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then b, c can be anything and $a = 3b + c$. The eigenspace is therefore all vectors of the form $\begin{bmatrix} 3b+c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. A basis for this space is $\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

For $\lambda = 0$, the eigenspace is the solutions to the linear system $\left[\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ -3 & 9 & 4 & 0 \end{array}\right]$. The RREF of this matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. The eigenspace is all vectors of the form $\begin{bmatrix} 1/3c \\ -1/3c \\ c \end{bmatrix}$ which has basis $\left\{\begin{bmatrix} 1/3 \\ -1/3 \\ 1 \end{bmatrix}\right\}$, or $\left\{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}\right\}$.

(b) $A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-1 & -3 & 0 & 0 \\ 0 & \lambda-2 & -4 & 0 \\ 0 & 0 & \lambda-2 & -3 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}\right) = (\lambda-1)^2(\lambda-2)^2$.

The eigenvalues are 1 and 2.

For $\lambda = 1$, the eigenspace is the solutions to $\left[\begin{array}{cccc|c} 0 & -3 & 0 & 0 & 0 \\ 0 & -1 & -4 & 0 & 0 \\ 0 & 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. This has RREF $\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ so the eigenspace is all vectors of the form $\begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and a basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 2$, the eigenspace is the solutions to $\left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$. This has RREF $\left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ so the eigenspace is all vectors of the form $\begin{bmatrix} 3b \\ b \\ 0 \\ 0 \end{bmatrix}$ and a basis is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

4. Suppose A is an invertible $n \times n$ matrix and λ is an eigenvalue of A. Prove that $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of A^{-1}.

   As λ is an eigenvalue of A, there exists $\mathbf{v} \neq \mathbf{0}$ with $A\mathbf{v} = \lambda\mathbf{v}$. If $\lambda = 0$, the equation $A\mathbf{v} = \lambda\mathbf{v}$ becomes $A\mathbf{v} = \mathbf{0}$ for some $\mathbf{v} \neq \mathbf{0}$. This cannot happen because A is invertible, so $\lambda \neq 0$. Also, if we take the equation $A\mathbf{v} = \lambda\mathbf{v}$ and multiply both sides of the equation by A^{-1} on the right, this equation becomes $\mathbf{v} = \lambda A^{-1}\mathbf{v}$. As $\lambda \neq 0$, we can divide by λ to get $A^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}$. The vector $\mathbf{v}$ is nonzero so this shows that $1/\lambda$ is an eigenvalue of A^{-1}.

5. Let A be a matrix with eigenvalues $\lambda_1 \neq \lambda_2$. Let W_1 be the eigenspace associated with λ_1 and W_2 be the eigenspace associated with λ_2. Prove that $W_1 \cap W_2 = \{\mathbf{0}\}$ (i.e. that the only vector in both eigenspaces is the zero vector).

   Suppose that $\mathbf{w}$ is a vector in both W_1 and W_2. As $\mathbf{w}$ is in W_1, we get that $A\mathbf{w} = \lambda_1\mathbf{w}$. Similarly, as $\mathbf{w}$ is in W_2, we get that $A\mathbf{w} = \lambda_2\mathbf{w}$. Then $\lambda_1\mathbf{w} = \lambda_2\mathbf{w}$ (because these are both equal to $A\mathbf{w}$). Then as $\lambda_1 \neq \lambda_2$, the equation $\lambda_1\mathbf{w} = \lambda_2\mathbf{w}$ forces $\mathbf{w} = \mathbf{0}$. This shows that the zero vector is the only vector which could possibly be in both W_1 and W_2. As $\mathbf{0}$ is in every subspace, it is in both W_1 and W_2 so $W_1 \cap W_2 = \{\mathbf{0}\}$.

6. Determine if the following statements are true or false. Give a proof or a counterexample.

   (a) If -5 is an eigenvalue of A, then 25 is an eigenvalue of A^2.

   True. If -5 is an eigenvalue of A, then there is some vector $\mathbf{v} \neq \mathbf{0}$ with $A\mathbf{v} = -5\mathbf{v}$. Then $A^2\mathbf{v} = A(A\mathbf{v}) = A(-5\mathbf{v}) = -5(A\mathbf{v}) = -5(-5\mathbf{v}) = 25\mathbf{v}$ so 25 is an eigenvalue of A^2.

   (b) If A and B are similar matrices and $\mathbf{x}$ is an eigenvector of A, then $\mathbf{x}$ is an eigenvector of B.

   False. Take for example $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$. These are the matrices from HW 13 problem 5 and they are similar as $B = P^{-1}AP$ with $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The matrix A has eigenvalue 0 with associated eigenvector $\mathbf{x} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ as $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This vector is not an eigenvector of B because $B\mathbf{x} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ -5 \end{bmatrix}$ which is not a multiple of $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$.