1. Let $L : P_3 \to M_{22}$ be the linear transformation given by $L(at^3 + bt^2 + ct + d) = \begin{bmatrix} a+b & c-d \\ 2c & 3d \end{bmatrix}$. Find representation of L with respect to the bases S and T where $S = \{t^3+t^2, t^2+t, t+1, 1\}$ and $T = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right\}$.

   Start by plugging the vectors from S into L. $L(t^3 + t^2) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $L(t^2 + t) = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, $L(t + 1) = \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}$, and $L(1) = \begin{bmatrix} 0 & -1 \\ 0 & 3 \end{bmatrix}$. For each of the resulting matrices, find the coordinate vector with respect to T. The first matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = 1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so its coordinate vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. The second is $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = 0 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so its coordinate vector is $\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. The third is $\begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} = -1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so its coordinate vector is $\begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$. The fourth is $\begin{bmatrix} 0 & -1 \\ 0 & 3 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so its coordinate vector is $\begin{bmatrix} -1/2 \\ 0 \\ 3 \\ 1/2 \end{bmatrix}$.

   The four coordinate vectors are the columns of the representation so the representation of L with respect to S and T is $\begin{bmatrix} 1 & 0 & -1 & -1/2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 1 & 1/2 \end{bmatrix}$.

2. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2a + c \\ c - b \end{bmatrix}$.

   Let S be the standard basis for $\mathbb{R}^3$ and T be the standard basis for $\mathbb{R}^2$. Let

$S' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \right\}$ and $T' = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

(a) Find the representation of L with respect to S and T.

First plug the vectors from S into L. $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, and $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Next take the coordinate vectors with respect to T. As T is the standard basis for $\mathbb{R}^2$, taking the coordinate vectors will not change these vectors so the representation of L with respect to S and T is $\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

(b) Find the representation of L with respect to S' and T' using the methods of Section 6.3.

First plug the vectors from S' into L. $L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $L\left(\begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $L\left(\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$. Next find coordinates with respect to T'. The first vector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ so the coordinate vector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The second vector is $\begin{bmatrix} 2 \\ 6 \end{bmatrix} = 0\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ so the coordinate vector is $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. The third vector is $\begin{bmatrix} 7 \\ 5 \end{bmatrix} = -4\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ so the coordinate vector is $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$. The coordinate vectors are the columns of the representation so the representation of L with respect to S' and T' is $\begin{bmatrix} -2 & 0 & -4 \\ 1 & 2 & 3 \end{bmatrix}$.

(c) Find the transition matrices P from S' to S and Q from T to T'.

To find the transition matrix P from S' to S, we take the vectors in S' and find their coordinates with respect to S. As S is the standard basis, taking coordinates does not change the vectors so the matrix is just

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -6 & -2 \\ 1 & 0 & 3 \end{bmatrix}.$$

To find the transition matrix Q from T to T', we take the vectors in T and find their coordinates with respect to T'. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{3}{4}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ so the coordinate vector is $\begin{bmatrix} -3/4 \\ 1/4 \end{bmatrix}$. $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ so the coordinate vector is $\begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}$. The matrix is $Q = \begin{bmatrix} -3/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$.

Another way to find Q is to first find the transition matrix from T' to T which is just $\begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$. This will be the inverse of Q so we can then find Q by computing the inverse of this matrix.

(d) Use your answers to parts (a) and (c) to find the representation of L with respect to S' and T' (as in Section 6.5). Check that this matches your answer to part (b).

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ be the representation of L with respect to S and T that we found in part (a). If B is the representation of L with respect to S' and T' then Section 6.5 tells us that $B = QAP$ where Q, P are the transition matrices found in part (c). We multiply this out to get $B = \frac{1}{4}\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 0 & -6 & -2 \\ 1 & 0 & 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 7 \\ 1 & 6 & 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -8 & 0 & -16 \\ 4 & 8 & 12 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -4 \\ 1 & 2 & 3 \end{bmatrix}$. This matches the answer to part (b).

3. Let $L : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 2c \\ 3a+b \\ b-d \\ d+a \end{bmatrix}$. Let S be the standard basis for $\mathbb{R}^4$ and $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Find the representation of L with respect to S.

Plug in the vectors from S into L. $L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\3\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$, $L\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\\0\\0\end{bmatrix}$, and $L\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$. Taking their coordinates with respect to S doesn't change the vectors so the representation of L with respect to S is $\begin{bmatrix}0&0&2&0\\3&1&0&0\\0&1&0&-1\\1&0&0&1\end{bmatrix}$.

(b) Find the representation of L with respect to T using the methods of Section 6.3.

Plug in the vectors from T into L. $L\left(\begin{bmatrix}1\\1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\4\\0\\2\end{bmatrix}$, $L\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\3\\0\\1\end{bmatrix}$, $L\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\\0\\0\end{bmatrix}$, and $L\left(\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\0\\-1\\1\end{bmatrix}$. Next find the coordinates of each of these vectors with respect to T. The first vector is $\begin{bmatrix}0\\4\\0\\2\end{bmatrix} = 4\begin{bmatrix}1\\1\\0\\1\end{bmatrix} - 4\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ so the coordinate vector is $\begin{bmatrix}4\\-4\\2\\-2\end{bmatrix}$. The second vector is $\begin{bmatrix}0\\3\\0\\1\end{bmatrix} = 3\begin{bmatrix}1\\1\\0\\1\end{bmatrix} - 3\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ so the coordinate vector is $\begin{bmatrix}3\\-3\\2\\-2\end{bmatrix}$. The third vector is $\begin{bmatrix}2\\0\\0\\0\end{bmatrix} = 0\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + 2\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ so the coordinate vector is $\begin{bmatrix}0\\2\\0\\0\end{bmatrix}$. The fourth vector is $\begin{bmatrix}2\\0\\-1\\1\end{bmatrix} = 0\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + 2\begin{bmatrix}1\\0\\0\\0\end{bmatrix} - 2\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ so the coordinate vector is $\begin{bmatrix}0\\2\\-2\\1\end{bmatrix}$. These are the columns of the representation so the representation of L with respect to T is $\begin{bmatrix}4 & 3 & 0 & 0\\-4 & -3 & 2 & 2\\2 & 2 & 0 & -2\\-2 & -2 & 0 & 1\end{bmatrix}$.

(c) Find the transitions matrices from S to T and from T to S.

As S is the standard basis, it is easier to compute the transition matrix from T to S. Let P be the transition matrix from T to S. This is the matrix whose columns are the vectors in T's coordinates with respect to S. Taking coordinates with respect to S doesn't change the vector so the columns of P are the vectors from T so $P = \begin{bmatrix}1 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 1 & 1\\1 & 0 & 0 & 1\end{bmatrix}$.

The transition matrix from S to T is P^{-1}. This can be computed either by taking $[P : I]$ and doing row operations to get $[I : P^{-1}]$ or by taking each vector from S and finding its coordinate with respect to T. Using the first method, the row operations $r_1 - r_2 \to r_1, r_4 - r_2 \to r_4, r_3 - r_4 \to r_3, r_1 \leftrightarrow r_2$ will take $[P : I]$ to $[I : P^{-1}]$. The matrix is $P^{-1} = \begin{bmatrix}0 & 1 & 0 & 0\\1 & -1 & 0 & 0\\0 & 1 & 1 & -1\\0 & -1 & 0 & 1\end{bmatrix}$.

(d) Use your answers to parts (a) and (c) to find the representation of L with respect to T (as in Section 6.5). Check that this matches your answer to part (b).

Let $A = \begin{bmatrix}0 & 0 & 2 & 0\\3 & 1 & 0 & 0\\0 & 1 & 0 & -1\\1 & 0 & 0 & 1\end{bmatrix}$ be the representation of L with respect to S found in part (a). Let B be the representation of L with respect to T. Then $B = P^{-1}AP$ where P is the transition matrix from T to S and P^{-1} is the transition matrix from S to T (found in part (b)). So $B =$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 2 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 1 & 0 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} 4 & 3 & 0 & 0 \\ -4 & -3 & 2 & 2 \\ 2 & 2 & 0 & -2 \\ -2 & -2 & 0 & 1 \end{bmatrix}.$$ This matches part (b).

4. Prove that if A and B are similar matrices then A^k and B^k are similar matrices for any positive integer k.

   If A and B are similar, then $B = P^{-1}AP$ for some invertible matrix P. Then $B^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}AIAP = P^{-1}A^2P$ so B^2 and A^2 are similar. In general, $B^k = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)$ multiplied k times. The PP^{-1} terms in the middle cancel and we get that $B^k = P^{-1}A^kP$ so B^k and A^k are similar.

5. Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The matrix P is invertible with inverse $P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Let $B = P^{-1}AP$.

   (a) Compute B.

   $$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

   (b) Do A and B have the same rank?

   Yes, they are both rank 2.

   (c) Do A and B have the same nullity?

   Yes, they both have nullity 1.

   (d) Do A and B have the same row space?

No. To show that two vector spaces are not the same space, we just need to find one vector that is in one but not the other. The vector $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$ is in the row space of B, as it is the first row of B. However it is not in the row space of A because it is not a linear combination of $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$ which are a basis for the row space of A.

(e) Do A and B have the same column space?

No. Anything in the column space of A has third entry 0, which is not true of all the columns of B.

(f) Do A and B have the same null space?

No. The null space of A is all multiples of $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$. The vector $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is in the null space of B but is not a multiple of $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ so is not in the null space of A.

6. If A and B are similar $n \times n$ matrices, which of the following are the same for A and B: rank, nullity, row space, column space, and null space?

The rank, and nullity must be the same but the row space, column space, and null space do not have to be the same.

The previous problem is a counter example that shows that the row space, column space, and null space of similar matrices do not have to be the same.

The rank of the two matrices must be equal - see Theorem 6.15 in the textbook. Note that similar matrices must be the same size. Therefore as the rank is the same and rank plus nullity is equal to the number of columns, the nullity must also be the same.