Homework 13 Due: Monday, November 17

1. Let $L : P_3 \to M_{22}$ be the linear transformation given by $L(at^3 + bt^2 + ct + d) = \begin{bmatrix} a+b & c-d \\ 2c & 3d \end{bmatrix}$. Find the representation of L with respect to the bases S and T where $S = \{t^3+t^2, t^2+t, t+1, 1\}$ and $T = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right\}$.

2. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2a+c \\ c-b \end{bmatrix}$. Let S be the standard basis for $\mathbb{R}^3$ and T be the standard basis for $\mathbb{R}^2$. Let $S' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \right\}$ and $T' = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

   (a) Find the representation of L with respect to S and T.

   (b) Find the representation of L with respect to S' and T' using the methods of Section 6.3.

   (c) Find the transition matrices P from S' to S and Q from T to T'.

   (d) Use your answers to parts (a) and (c) to find the representation of L with respect to S' and T' (as in Section 6.5). Check that this matches your answer to part (b).

3. Let $L : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} 2c \\ 3a+b \\ b-d \\ d+a \end{bmatrix}$. Let S be the standard basis for $\mathbb{R}^4$ and $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

   (a) Find the representation of L with respect to S.

   (b) Find the representation of L with respect to T using the methods of Section 6.3.

   (c) Find the transitions matrices from S to T and from T to S.

   (d) Use your answers to parts (a) and (c) to find the representation of L with respect to T (as in Section 6.5). Check that this matches your answer to part (b).

4. Prove that if A and B are similar matrices then A^k and B^k are similar matrices for any positive integer k.

5. Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The matrix P is invertible with inverse $P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Let $B = P^{-1}AP$.

   (a) Compute B.

   (b) Do A and B have the same rank?

   (c) Do A and B have the same nullity?

   (d) Do A and B have the same row space?

   (e) Do A and B have the same column space?

   (f) Do A and B have the same null space?

6. If A and B are similar $n \times n$ matrices, which of the following are the same for A and B: rank, nullity, row space, column space, and null space?