1. Which of the following are linear transformations?

(a) $L : \mathbb{R}_2 \to M_{22}$ defined by $L\left(\begin{bmatrix} a & b \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ b & a+b \end{bmatrix}$.

This is a linear transformation. If $\begin{bmatrix} a_1 & b_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \end{bmatrix}$ are two vectors in $\mathbb{R}_2$, then $L\left(\begin{bmatrix} a_1 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \end{bmatrix}\right)$

$$= \begin{bmatrix} a_1+a_2 & 0 \\ b_1+b_2 & a_1+a_2+b_1+b_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ b_1 & a_1+b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ b_2 & a_2+b_2 \end{bmatrix}$$

$= L\left(\begin{bmatrix} a_1 & b_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \end{bmatrix}\right)$. If $\begin{bmatrix} a & b \end{bmatrix}$ is a vector in $\mathbb{R}_2$ and r is a real number then $L\left(r\begin{bmatrix} a & b \end{bmatrix}\right) = L\left(\begin{bmatrix} ra & rb \end{bmatrix}\right) = \begin{bmatrix} ra & 0 \\ rb & ra+rb \end{bmatrix} = r\begin{bmatrix} a & 0 \\ b & a+b \end{bmatrix} = rL\left(\begin{bmatrix} a & b \end{bmatrix}\right)$.

(b) $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ b+1 \\ a \end{bmatrix}$.

This is not a linear transformation. It fails both of the properties of a linear transformation. To show it is not a linear transformation, you only need to show that it fails one of the properties, but here we will check both. If $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$ are two vectors in $\mathbb{R}^2$, then $L\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} a_1+a_2 \\ b_1+b_2 \end{bmatrix}\right) = \begin{bmatrix} a_1+a_2-(b_1+b_2) \\ b_1+b_2+1 \\ a_1+a_2 \end{bmatrix}$ and $L\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) = \begin{bmatrix} a_1-b_1 \\ b_1+1 \\ a_1 \end{bmatrix} + \begin{bmatrix} a_2-b_2 \\ b_2+1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2-(b_1+b_2) \\ b_1+b_2+2 \\ a_1+a_2 \end{bmatrix}$. If $\begin{bmatrix} a \\ b \end{bmatrix}$ is a vector in $\mathbb{R}^2$ and r is a real number then $L\left(r\begin{bmatrix} a \\ b \end{bmatrix}\right) = L\left(\begin{bmatrix} ra \\ rb \end{bmatrix}\right) = \begin{bmatrix} ra-rb \\ rb+1 \\ ra \end{bmatrix}$ and $rL(\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = r\begin{bmatrix} a-b \\ b+1 \\ a \end{bmatrix} = \begin{bmatrix} ra-rb \\ rb+r \\ ra \end{bmatrix}$. For both properties, they fail because of the middle component $b+1$.

(c) $L : P_3 \to P_2$ defined by $L(p(t)) = p'(t)$.

This is a linear transformation. Note that if $p(t)$ has degree at most 3, then $p'(t)$ has degree at most 2. If $p(t), q(t)$ are in P_3 then $L(p(t)+q(t)) = (p(t)+q(t))' = p'(t)+q'(t) = L(p(t)) + L(q(t))$. If $p(t)$ is in P_3 and r is a real number, $L(rp(t)) = (rp(t))' = rp'(t) = rL(p(t))$.

2. Let L be the linear transformation $L : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+2d \\ d+b+c \\ c \\ b-a \end{bmatrix}$. Find the standard matrix representing L.

To find this matrix, take the vectors from the standard basis for $\mathbb{R}^4$ and plug them into L. $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. These are the columns of the standard matrix representing L so the matrix is $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$.

3. Suppose L is a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that $L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$.

(a) Find $L\left(\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}\right)$.

First write $\begin{bmatrix}1\\3\\-2\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$. This is $\begin{bmatrix}1\\3\\-2\end{bmatrix} = -2\begin{bmatrix}1\\1\\1\end{bmatrix} + 5\begin{bmatrix}1\\1\\0\end{bmatrix} - 2\begin{bmatrix}1\\0\\0\end{bmatrix}$. Then $L\left(\begin{bmatrix}1\\3\\-2\end{bmatrix}\right) = L\left(-2\begin{bmatrix}1\\1\\1\end{bmatrix} + 5\begin{bmatrix}1\\1\\0\end{bmatrix} - 2\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$ and by the properties of linear transformations this is $L\left(-2\begin{bmatrix}1\\1\\1\end{bmatrix}\right) + L\left(5\begin{bmatrix}1\\1\\0\end{bmatrix}\right) + L\left(-2\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = -2L\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) + 5L\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) - 2L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = -2\begin{bmatrix}1\\0\\-1\end{bmatrix} + 5\begin{bmatrix}2\\2\\1\end{bmatrix} - 2\begin{bmatrix}0\\-2\\5\end{bmatrix} = \begin{bmatrix}8\\14\\-3\end{bmatrix}$.

(b) Find a general formula for $L\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right)$.

We follow the same process as in the previous part, but for a general vector $\begin{bmatrix}a\\b\\c\end{bmatrix}$ instead of the specific vector given in (a). $\begin{bmatrix}a\\b\\c\end{bmatrix} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + (b-c)\begin{bmatrix}1\\1\\0\end{bmatrix} + (a-b)\begin{bmatrix}1\\0\\0\end{bmatrix}$ so $L\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = cL\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) + (b-c)\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) + (a-b)L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = c\begin{bmatrix}1\\0\\-1\end{bmatrix} + (b-c)\begin{bmatrix}2\\2\\1\end{bmatrix} + (a-b)\begin{bmatrix}0\\-2\\5\end{bmatrix} = \begin{bmatrix}2b-c\\-2a+4b-2c\\5a-4b-2c\end{bmatrix}$.

4. Let $L : M_{22} \to P_3$ be the linear transformation given by

$$L\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = (a+b)t^3 + (a-c)t^2 + (2a-d)t + (2a+b-c)\ .$$

(a) Find a basis for $\ker L$.

The kernel of L is all matrices $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ such that $a+b=0, a-c=0, 2a-d=0, 2a+b-c=0$. We can take a to be anything and then take $b=-a, c=a, d=2a$. The kernel is therefore all matrices of the form $\begin{bmatrix}a & -a\\a & 2a\end{bmatrix}$. This has basis $\left\{\begin{bmatrix}1 & -1\\1 & 2\end{bmatrix}\right\}$.

(b) Find a basis for range L.

The range of L is all polynomials of the form $(a+b)t^3+(a-c)t^2+(2a-d)t+(2a+b-c) = a(t^3+t^2+2t+2)+b(t^3+1)+c(-t^2-1)+d(-t)$. The range is thus spanned by $\{t^3+t^2+2t+2, t^3+1, -t^2-1, -t\}$. Some subset of this set will be a basis for the range. It may be helpful at this point to compute the dimension of the range of L so that we know how many vectors we will need to delete to get a basis. For a linear transformation $L : V \to W$, the dimensions satisfy the formula $\dim V = \dim \ker L + \dim \text{range } L$. Here $V = M_{22}$ which has dimension 4 and from part (a), the kernel has dimension 1. It follows that $\dim \text{range } L = 3$ so we will need to delete one of the four vectors in the spanning set to get a basis. In this case, any one of the four vectors can be written as a linear combination of the others so any one of them can be deleted. For example, $t^3+t^2+2t+2 = 1(t^3+1)+(-1)(-t^2-1)+(-2)(-t)$ so the first is a linear combination of the others and one possible basis would be $\{t^3 + 1, -t^2 - 1, -t\}$.

(c) Is L one-to-one? Is L onto?

L is not one-to-one since its kernel is has dimension 1, not dimension 0. It is also not onto because the range has dimension 3 and P_3 had dimension 4 so the range is not all of P_3.

5. Let $L : M_{33} \to M_{33}$ be defined by $L(A) = A - A^T$.

(a) Prove that L is a linear transformation.

Check the two properties of linear transformations. If A, B are in M_{33} then $L(A+B) = (A+B)-(A+B)^T = A+B-(A^T+B^T) = A+B-A^T-B^T = (A-A^T)+(B-B^T) = L(A)+L(B)$. If A is in M_{33} and r is a real number, $L(rA) = (rA) - (rA)^T = rA - rA^T = r(A - A^T) = rL(A)$.

(b) Describe the matrices in the kernel of L. What is $\dim \ker L$?

If A is in the kernel of L then $\mathbf{0} = L(A) = A - A^T$ so $A = A^T$. The kernel of L is exactly the set of all 3×3 symmetric matrices. The symmetric matrices form a subspace of dimension 6 so $\dim \ker L = 6$.

Note: The dimension is 6 because the symmetric matrices have the form

$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ and have basis

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

(c) Prove that the range of L is the set of all 3×3 skew symmetric matrices. What is $\dim \operatorname{range} L$?

The range of L is all matrices of the form $A - A^T$. Matrices of this form are skew symmetric as $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$. This shows that the range of L is contained in the set of all 3×3 skew symmetric matrices and hence the range of L is a subspace of the space of 3×3 skew symmetric matrices. The skew symmetric matrices have the form $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ and have basis $\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$. Hence the set of all skew symmetric matrices is a subspace of M_{33} with dimension 3. The dimension of the range of L is $\dim \operatorname{range} L = \dim M_{33} - \dim \ker L = 9 - 6 = 3$. The range of L is a subspace of the skew symmetric matrices and both spaces have dimension 3, so they must be the same space.

6. Let A be an $m \times n$ matrix. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by $L(\mathbf{v}) = A\mathbf{v}$. Show that $\dim \ker L$ is equal to the nullity of A and $\dim \operatorname{range} L$ is equal to the rank of A.

A vector $\mathbf{v}$ is in the kernel of L if and only if $\mathbf{0} = L(\mathbf{v}) = A\mathbf{v}$. It follows that the kernel of L is exactly the null space of A so $\dim \ker L$ is the nullity of A. The dimension of the range is $\dim \operatorname{range} L = \dim \mathbb{R}^n - \dim \ker L$ which is n minus the nullity of A. Also, the rank plus nullity of A equal the number of columns which is n so the rank is also n minus the nullity of A. The rank of A is therefore equal to the dimension of the range of L.

Note: Another way to prove the statement about the range is to show that the range of L is the same as the column space of A. This is true because if $c_1, c_2, ..., c_n$ are the columns of A and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then $A\mathbf{v} = v_1c_2 + v_2c_2 + ... + v_nc_n$.

7. Let V be an inner product space and $\mathbf{w_1}, \mathbf{w_2}$ be nonzero orthogonal vectors in V. Let $L : V \to \mathbb{R}^2$ be defined by $L(\mathbf{v}) = \begin{bmatrix} (\mathbf{v}, \mathbf{w_1}) \\ (\mathbf{v}, \mathbf{w_2}) \end{bmatrix}$.

(a) Show that L is a linear transformation.

Let $\mathbf{v_1}, \mathbf{v_2}$ be two vectors in V. Then $L(\mathbf{v_1} + \mathbf{v_2}) = \begin{bmatrix} (\mathbf{v_1} + \mathbf{v_2}, \mathbf{w_1}) \\ (\mathbf{v_1} + \mathbf{v_2}, \mathbf{w_2}) \end{bmatrix} = \begin{bmatrix} (\mathbf{v_1}, \mathbf{w_1}) + (\mathbf{v_2}, \mathbf{w_1}) \\ (\mathbf{v_1}, \mathbf{w_2}) + (\mathbf{v_2}, \mathbf{w_2}) \end{bmatrix} = \begin{bmatrix} (\mathbf{v_1}, \mathbf{w_1}) \\ (\mathbf{v_1}, \mathbf{w_2}) \end{bmatrix} + \begin{bmatrix} (\mathbf{v_2}, \mathbf{w_1}) \\ (\mathbf{v_2}, \mathbf{w_2}) \end{bmatrix} = L(\mathbf{v_1}) + L(\mathbf{v_2})$. Note that this uses property 3 of inner products. If $\mathbf{v}$ is in V and r is a real number then $L(r\mathbf{v}) = \begin{bmatrix} (r\mathbf{v}, \mathbf{w_1}) \\ (r\mathbf{v}, \mathbf{w_2}) \end{bmatrix} = \begin{bmatrix} r(\mathbf{v}, \mathbf{w_1}) \\ r(\mathbf{v}, \mathbf{w_2}) \end{bmatrix} = r \begin{bmatrix} (\mathbf{v}, \mathbf{w_1}) \\ (\mathbf{v}, \mathbf{w_2}) \end{bmatrix} = rL(\mathbf{v})$. This uses property 4 of inner products.

(b) Show that L is onto. Hint: Consider $L(\mathbf{w_1})$ and $L(\mathbf{w_2})$.

As $\mathbf{w_1}, \mathbf{w_2}$ are orthogonal, $(\mathbf{w_1}, \mathbf{w_2}) = (\mathbf{w_2}, \mathbf{w_1}) = 0$. Then $L(\mathbf{w_1}) = \begin{bmatrix} (\mathbf{w_1}, \mathbf{w_1}) \\ (\mathbf{w_1}, \mathbf{w_2}) \end{bmatrix} = \begin{bmatrix} (\mathbf{w_1}, \mathbf{w_1}) \\ 0 \end{bmatrix}$ and $L(\mathbf{w_2}) = \begin{bmatrix} (\mathbf{w_2}, \mathbf{w_1}) \\ (\mathbf{w_2}, \mathbf{w_2}) \end{bmatrix} = \begin{bmatrix} 0 \\ (\mathbf{w_2}, \mathbf{w_2}) \end{bmatrix}$. Note that $\mathbf{w_1}, \mathbf{w_2}$ are nonzero so by property 1 of inner products, $(\mathbf{w_1}, \mathbf{w_1}) \neq 0$ and $(\mathbf{w_2}, \mathbf{w_2}) \neq 0$. It follows that the vectors $\left\{ \begin{bmatrix} (\mathbf{w_1}, \mathbf{w_1}) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ (\mathbf{w_2}, \mathbf{w_2}) \end{bmatrix} \right\}$ are a basis for $\mathbb{R}^2$. These are both in the range of L and the range of L is a subspace of $\mathbb{R}^2$, so the range of L must be all of $\mathbb{R}^2$.

(c) If $\dim V = n$, what is $\dim \ker L$?

$\dim \ker L = \dim V - \dim \text{range } L = n - 2$.

(d) Let $W = \text{span}\{\mathbf{w_1}, \mathbf{w_2}\}$. How are $\ker L$ and $W^\perp$ related?

The kernel of L is exactly the set of vectors which are orthogonal to both $\mathbf{w_1}, \mathbf{w_2}$. A vector $\mathbf{v}$ is orthogonal to $\mathbf{w_1}$ and $\mathbf{w_2}$ if and only if it is orthogonal to everything in their span which is W. It follows that $\ker L = W^\perp$.

8. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Show that L is invertible and find L^{-1}.

The range of L contains $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$ which is a basis for $\mathbb{R}^2$. The range is therefore all of $\mathbb{R}^2$ and L is onto. Then $\dim \ker L = \dim \mathbb{R}^2 - \dim \operatorname{range} L = 2-2 = 0$ so L is also one-to-one. It is one-to-one and onto so it is invertible.

To find L^{-1}, use that $L^{-1}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $L^{-1}\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. To find out what L^{-1} does to a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ we must figure out a way to write it as a linear combination of $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$. If we write $\begin{bmatrix} a \\ b \end{bmatrix} = x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ this is the linear system $\left[\begin{array}{cc|c} 2 & 5 & a \\ 1 & 3 & b \end{array}\right]$. The row operations $r_1 \leftrightarrow r_2, r_2 - 2r_1 \to r_2, -r_2 \to r_2, r_1 - 3r_2 \to r_2$ change this to RREF which is $\left[\begin{array}{cc|c} 1 & 0 & 3a-5b \\ 0 & 1 & 2b-a \end{array}\right]$. This tells us that $\begin{bmatrix} a \\ b \end{bmatrix} = (3a-5b)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (2b-a)\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ so $L^{-1}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = L^{-1}\left((3a-5b)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (2b-a)\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right) = (3a-5b)L^{-1}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) + (2b-a)L^{-1}\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right) = (3a-5b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (2b-a)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3a-5b \\ 2b-a \end{bmatrix}$.

Another way to do this is to use the standard matrix representing L. In this case, the matrix is $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$. This matrix is invertible because $\det(A) = 1 \neq 0$ so L is also invertible. Its inverse will be the linear transformation whose standard matrix representing it is A^{-1}. We can compute A^{-1} either using the formula for 2×2 inverses, or by writing $[A : I]$ and doing row operations to get $[I : A^{-1}]$. The same row operations as above will take $[A : I]$ to $[I : A^{-1}]$ so $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$. Then $L^{-1}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a-5b \\ -a+2b \end{bmatrix}$.

9. Let A be an invertible $n \times n$ matrix and let $L : M_{nn} \to M_{nn}$ be defined by $L(B) = A^{-1}BA$.

   (a) Show that L is a linear transformation.

   If B, C are $n \times n$ matrices, then $L(B+C) = A^{-1}(B+C)A = A^{-1}BA + A^{-1}CA = L(B) + L(C)$ and if B is and $n \times n$ matrix and r is a real number then $L(rB) = A^{-1}(rB)A = r(A^{-1}BA) = rL(B)$.

(b) Find $\ker L$ and range L.

If B is in $\ker L$ then $A^{-1}BA = \mathbf{0}$. Multiplying on the left of both sides by A this equation becomes $AA^{-1}BA = A\mathbf{0}$ which simplifies to $BA = \mathbf{0}$. Multiplying on the right of both sides by A^{-1} (because we are given that A is invertible), gives $BAA^{-1} = \mathbf{0}A^{-1}$ which simplifies to $B = \mathbf{0}$. Therefore the only thing in $\ker L$ is $\mathbf{0}$ so $\ker L = \{\mathbf{0}\}$. Then $\dim \text{range } L = \dim M_{nn} - \dim \ker L = n^2 - 0 = n^2$. As range L is a subspace of M_{nn} and they have the same finite dimension, range $L = M_{nn}$.

(c) Is L invertible? If yes, what is L^{-1}?

Yes. L is one-to-one since $\ker L = \{\mathbf{0}\}$ and L is onto because range $L = M_{nn}$. L is both one-to-one and onto so it is invertible. The inverse of L is $L^{-1}(B) = ABA^{-1}$. We check this as follows: $L(L^{-1}(B)) = L(ABA^{-1}) = A^{-1}(ABA^{-1})A = (A^{-1}A)B(A^{-1}A) = B$ and $L^{-1}(L(B)) = L^{-1}(A^{-1}BA) = A(A^{-1}BA)A^{-1} = (AA^{-1})B(AA^{-1}) = B$.