Homework 12 Due: Monday, November 10

1. Which of the following are linear transformations?

(a) $L : \mathbb{R}_2 \to M_{22}$ defined by $L\left(\begin{bmatrix} a & b \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ b & a+b \end{bmatrix}$.

(b) $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-b \\ b+1 \\ a \end{bmatrix}$.

(c) $L : P_3 \to P_2$ defined by $L(p(t)) = p'(t)$.

2. Let L be the linear transformation $L : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $L\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+2d \\ d+b+c \\ c \\ b-a \end{bmatrix}$. Find the standard matrix representing L.

3. Suppose L is a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that $L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$.

(a) Find $L\left(\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}\right)$.

(b) Find a general formula for $L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$.

4. Let $L : M_{22} \to P_3$ be the linear transformation given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)t^3 + (a-c)t^2 + (2a-d)t + (2a+b-c) .$$

(a) Find a basis for $\ker L$.

(b) Find a basis for range L.

(c) Is L one-to-one? Is L onto?

5. Let $L : M_{33} \to M_{33}$ be defined by $L(A) = A - A^T$.

   (a) Prove that L is a linear transformation.

   (b) Describe the matrices in the kernel of L. What is $\dim \ker L$?

   (c) Prove that the range of L is the set of all 3×3 skew symmetric matrices. What is $\dim \operatorname{range} L$?

6. Let A be an $m \times n$ matrix. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by $L(\mathbf{v}) = A\mathbf{v}$. Show that $\dim \ker L$ is equal to the nullity of A and $\dim \operatorname{range} L$ is equal to the rank of A.

7. Let V be an inner product space and $\mathbf{w_1}, \mathbf{w_2}$ be nonzero orthogonal vectors in V. Let $L : V \to \mathbb{R}^2$ be defined by $L(\mathbf{v}) = \begin{bmatrix} (\mathbf{v}, \mathbf{w_1}) \\ (\mathbf{v}, \mathbf{w_2}) \end{bmatrix}$.

   (a) Show that L is a linear transformation.

   (b) Show that L is onto. Hint: Consider $L(\mathbf{w_1})$ and $L(\mathbf{w_2})$.

   (c) If $\dim V = n$, what is $\dim \ker L$?

   (d) Let $W = \operatorname{span}\{\mathbf{w_1}, \mathbf{w_2}\}$. How are $\ker L$ and $W^\perp$ related?

8. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Show that L is invertible and find L^{-1}.

9. Let A be an invertible $n \times n$ matrix and let $L : M_{nn} \to M_{nn}$ be defined by $L(B) = A^{-1}BA$.

   (a) Show that L is a linear transformation.

   (b) Find $\ker L$ and $\operatorname{range} L$.

   (c) Is L invertible? If yes, what is L^{-1}?