For all problems involving $\mathbb{R}^n$, you may assume the inner product is the dot product unless otherwise specified.

1. Let $S = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$.

   Verify that S is an orthonormal basis for $\mathbb{R}^4$. Let $\mathbf{v} = \begin{bmatrix} 4 \\ -1 \\ 2 \\ 7 \end{bmatrix}$. Use dot products to find $[\mathbf{v}]_S$.

   All pairs of distinct vectors have dot product 0 (there are 6 pairs to check) and all vectors in S have length 1, so S is orthonormal. It is linearly independent because it is orthogonal. S is a linearly independent set of 4 vectors in a 4-dimensional vector space so it is a basis. To find $[\mathbf{v}]_S$, write $\mathbf{v} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} + a_4\mathbf{v_4}$ where $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ are the vectors in S. As S is orthonormal, $a_i = \mathbf{v_i} \cdot \mathbf{v}$. So we can compute the a_i by taking 4 dot products.

   $$a_1 = \mathbf{v_1} \cdot \mathbf{v} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \\ 7 \end{bmatrix} = 3/\sqrt{2}$$

   $$a_2 = \mathbf{v_2} \cdot \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \\ 7 \end{bmatrix} = 9/\sqrt{2}$$

   $$a_3 = \mathbf{v_3} \cdot \mathbf{v} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \\ 7 \end{bmatrix} = 0$$

   $$a_4 = \mathbf{v_4} \cdot \mathbf{v} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \\ 7 \end{bmatrix} = 5$$

   So $[\mathbf{v}]_S = \begin{bmatrix} 3/\sqrt{2} \\ 9/\sqrt{2} \\ 0 \\ 5 \end{bmatrix}$.

2. Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}$. S is a basis for $\mathbb{R}^3$. Use the Gram-Schmidt process to transform S into:

(a) an orthogonal basis.

Let $\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. The new basis will be $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

Take $\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Then $\mathbf{v_2} = -\frac{\mathbf{v_1} \cdot \mathbf{u_2}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} + \mathbf{u_2} = -\frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then $\mathbf{v_3} = -\frac{\mathbf{v_1} \cdot \mathbf{u_3}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} - \frac{\mathbf{v_2} \cdot \mathbf{u_3}}{\mathbf{v_2} \cdot \mathbf{v_2}} \mathbf{v_2} + \mathbf{u_3} = -\frac{(-1)}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$.

The orthogonal basis you get is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} \right\}$. You can also replace $\mathbf{v_3}$ with $2\mathbf{v_3}$ and it will still be an orthogonal basis and there won't be any fractions. The resulting basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$.

(b) an orthonormal basis.

Take the basis from (a) and divide each vector by its length. The resulting orthonormal basis is $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\}$.

3. Use the Gram-Schmidt process to find an orthonormal basis for W where W is the subspace of $\mathbb{R}^4$ which consists of all vectors of the form $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $a + b + c + d = 0$.

The answer to this problem depends on the basis you pick for W. One possibility is to solve for d so $d = -a - b - c$ then rewrite $\begin{bmatrix} a \\ b \\ c \\ -a-b-c \end{bmatrix}$ as $a\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. The basis is then $\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. Then $\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

Then $\mathbf{v_2} = -\frac{\mathbf{v_1}\cdot\mathbf{u_2}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} + \mathbf{u_2} = -\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ -1/2 \end{bmatrix}$. To avoid fractions, we can replace $\mathbf{v_2}$ with $2\mathbf{v_2}$, so instead take $\mathbf{v_2} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$

. Then $\mathbf{v_3} = -\frac{\mathbf{v_1}\cdot\mathbf{u_3}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{v_2}\cdot\mathbf{u_3}}{\mathbf{v_2}\cdot\mathbf{v_2}}\mathbf{v_2} + \mathbf{u_3} = -\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ -1/3 \end{bmatrix}$.

To avoid fractions, we can replace $\mathbf{v_3}$ with $3\mathbf{v_3}$ so $\mathbf{v_3} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$.

The orthogonal basis you get is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$. To get an orthonormal basis, divide each vector by its length.

The resulting basis is $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 2\sqrt{6} \\ 0 \\ -1\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \\ -1/\sqrt{12} \end{bmatrix} \right\}$.

Note: If you started by solving for a instead of d, the original basis would be $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and the orthogonal matrix resulting from Gram-Schmidt would be $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \right\}$. The resulting orthonormal basis would then be $\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{12} \\ -1/\sqrt{12} \\ -1/\sqrt{12} \\ 3/\sqrt{12} \end{bmatrix} \right\}$.

4. Let V be P_2 with inner product $(p(t), q(t)) = \int_0^1 p(t)q(t)\ dt$. Use the Gram-Schmidt process to transform the basis $\{1, t, t^2\}$ into an orthogonal basis.

   Let $\mathbf{u_1} = 1, \mathbf{u_2} = t, \mathbf{u_3} = t^2$. Take $\mathbf{v_1} = \mathbf{u_1} = 1$. Then $\mathbf{v_2} = -\frac{(\mathbf{v_1},\mathbf{u_2})}{(\mathbf{v_1},\mathbf{v_1})}\mathbf{v_1} + \mathbf{u_2}$. The inner products are $(\mathbf{v_1}, \mathbf{u_2}) = (1, t) = \int_0^1 (1)(t)\ dt = 1/2$ and $(\mathbf{v_1}, \mathbf{v_1}) = (1, 1) = \int_0^1 (1)(1)\ dt = 1$. Plugging these in we get that $\mathbf{v_2} = -\frac{(1/2)}{1}1 + t = t - \frac{1}{2}$. If we want to avoid fractions, we can replace this with $2t - 1$ so $\mathbf{v_2} = 2t - 1$. Then $\mathbf{v_3} = -\frac{(\mathbf{v_1},\mathbf{u_3})}{(\mathbf{v_1},\mathbf{v_1})}\mathbf{v_1} - \frac{(\mathbf{v_2},\mathbf{u_3})}{(\mathbf{v_2},\mathbf{v_2})}\mathbf{v_2} + \mathbf{u_3}$. The inner products are $(\mathbf{v_1}, \mathbf{u_3}) = (1, t^2) = \int_0^1 (1)(t^2)\ dt = 1/3$, $(\mathbf{v_1}, \mathbf{v_1}) = 1$, $(\mathbf{v_2}, \mathbf{u_3}) = (2t - 1, t^2) = \int_0^1 (2t - 1)(t^2)\ dt = \int_0^1 2t^3 - t^2\ dt = \frac{1}{2} - \frac{1}{3} = 1/6$, and $(\mathbf{v_2}, \mathbf{v_2}) = (2t - 1, 2t - 1) = \int_0^1 (2t - 1)^2\ dt = \int_0^1 4t^2 - 4t + 1\ dt = \frac{4}{3} - 2 + 1 = \frac{1}{3}$. Plugging these in we get $\mathbf{v_3} = -\frac{(1/3)}{1}1 - \frac{(1/6)}{(1/3)}(2t - 1) + t^2 = t^2 - t + \frac{1}{6}$, or to avoid fractions, $6t^2 - 6t + 1$.

   The orthogonal basis is $\{1, 2t - 1, 6t^2 - 6t + 1\}$.

5. Let W be the subspace of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \right\}$.

   (a) Find a basis for $W^\perp$.

   The is the set of all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ which are perpendicular to all vectors in W. Note that if a vector is perpendicular a set of vectors, then it is perpendicular to all the linear combinations of those vectors. It follows that we just need to find the vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ which are perpendicular to the three spanning vectors. If we take dot products, we get that $0 = \begin{bmatrix}1\\2\\4\end{bmatrix} \cdot \begin{bmatrix}a\\b\\c\end{bmatrix} = a + 2b + 4c$, $0 = \begin{bmatrix}-1\\2\\0\end{bmatrix} \cdot \begin{bmatrix}a\\b\\c\end{bmatrix} = -a + 2b$, $0 = \begin{bmatrix}3\\1\\7\end{bmatrix} \cdot \begin{bmatrix}a\\b\\c\end{bmatrix} = 3a+b+7c$. We are therefore looking for all solutions to the homogeneous linear system $a + 2b + 4c = 0, -a + 2b = 0, 3a + b + 7c = 0$. This has coefficient matrix $\begin{bmatrix}1 & 2 & 4\\-1 & 2 & 0\\3 & 1 & 7\end{bmatrix}$. The RREF of this matrix is $\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$. Then c can be anything, $b = -c, a = -2c$ so the solution are all vectors of the form $\begin{bmatrix}-2c\\-c\\c\end{bmatrix}$.

$W^\perp$ is the set of all vectors of the form $\begin{bmatrix}-2c\\-c\\c\end{bmatrix}$. These are all the multiples of $\begin{bmatrix}-2\\-1\\1\end{bmatrix}$ so $\left\{\begin{bmatrix}-2\\-1\\1\end{bmatrix}\right\}$ is a basis for $W^\perp$.

(b) Find $\dim W$ and $\dim W^\perp$.

By the pervious part, $\dim W^\perp = 1$ since a basis for $W^\perp$ has size 1. Then $\dim W + \dim W^\perp = \dim \mathbf{R}^3$ so $\dim W = 2$. Note that the given spanning set for W is not linearly independent as the last vector is a linear combination of the first two.

(c) Describe W and $W^\perp$ geometrically.

W is a 2 dimensional subspace of $\mathbb{R}^3$ so it is a plane through the origin. $W^\perp$ is a one dimensional subspace of $\mathbb{R}^3$ so it is a line through the origin. Note that the line $W^\perp$ is normal to the plane W.

6. Let $A = \begin{bmatrix}1 & 0 & 3 & 1 & 0\\0 & 1 & 0 & -1 & 2\\2 & 1 & 6 & 1 & 2\\0 & 0 & 1 & 0 & -3\end{bmatrix}$. Let U be the null space of A and W be the column space of A^T. Note that U and W are both subspaces of $\mathbb{R}^5$.

(a) Find a basis for U.

The RREF of A is $\begin{bmatrix} 1 & 0 & 0 & 1 & 9 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. If we use variables x, y, z, r, s then r, s are anything and $z = 3s, y = r - 2s, x = -r - 9s$. The null space is all vectors of the form $\begin{bmatrix} -r-9s \\ r-2s \\ 3s \\ r \\ s \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -9 \\ -2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$. A basis for the null space is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(b) Find a basis for W.

From the RREF of A, a basis for the row space of A is $\left\{ \begin{bmatrix} 1 & 0 & 0 & 1 & 9 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & -3 \end{bmatrix} \right\}$ so a basis for the column space of A^T will be $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -3 \end{bmatrix} \right\}$.

Another possible method to find a basis for the column space of A^T is to find RREF of A^T then take the columns of A^T that correspond to columns in RREF with leading ones. The basis you get from this method is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -3 \end{bmatrix} \right\}$.

(c) Show that if $\mathbf{u}$ is in U and $\mathbf{w}$ is in W then $\mathbf{u} \cdot \mathbf{w} = 0$.

If $\mathbf{u}$ is in U then $\mathbf{u} = \begin{bmatrix} -r-9s \\ r-2s \\ 3s \\ r \\ s \end{bmatrix}$ for some r, s. If $\mathbf{w}$ is in W then $\mathbf{w} =$

$\begin{bmatrix} x \\ y \\ z \\ x-y \\ 9x+2y-3z \end{bmatrix}$ for some x, y, z. Note that we are using the first basis found in part (b), it will also work to use the second basis. Then $\mathbf{u} \cdot \mathbf{w} = \begin{bmatrix} -r-9s \\ r-2s \\ 3s \\ r \\ s \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ x-y \\ 9x+2y-3z \end{bmatrix} = (-r-9s)(x)+(r-2s)(y)+(3s)(z)+(r)(x-y)+(s)(9x+2y-3z) = -rx-9sx+ry-2sy+3sz+rx-ry+9xs+2sy-3sz = 0$.

7. Let V be a finite dimensional inner product space and let W be a subspace of V. Find $\dim W^{\perp}$ if:

   (a) $\dim V = 7$ and $\dim W = 3$.
   $\dim W^{\perp} = \dim V - \dim W = 7 - 3 = 4$.

   (b) $V = \mathbb{R}^2$ and W is a line through the origin.

   $\dim W^{\perp} = \dim V - \dim W = 2 - 1 = 1$.

   (c) $V = \mathbb{R}^3$ and W is a line through the origin.

   $\dim W^{\perp} = \dim V - \dim W = 3 - 1 = 2$.