1. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. Use the dot product on $\mathbb{R}^3$ to compute the following.

   (a) The lengths of $\mathbf{u}$ and $\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

   $$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$
   $$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{3^2 + 0^2 + 1^2} = \sqrt{10}$$

   (b) The distance between $\mathbf{u}$ and $\mathbf{v}$.

   $$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(1-3)^2 + (2-0)^2 + (-1-1)^2} = \sqrt{12}$$

   (c) The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

   $$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2}{\sqrt{6}\sqrt{10}}$$

2. Determine if the following sets of vectors in $\mathbb{R}^4$ with the dot product are orthogonal, orthonormal, or neither.

   (a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$

   This is orthogonal as the dot product of any two of the three is 0, but the vectors are not length 1.

   (b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}$

   This is neither. The dot product of the last vector with any of the other vectors is not 0.

   (c) $\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}$

   This is orthonormal. The dot product of any two is 0 and they are all length 1.

3. Let $V = P$ with inner product $(p(t), q(t)) = \int_0^1 p(t)q(t)\ dt$.
Let $r(t) = t^4, s(t) = 3t^4 - 1$.

   (a) Find $\|r(t)\|$ and $\|s(t)\|$.

   $\|r(t)\| = \sqrt{\int_0^1 r(t)^2\ dt} = \sqrt{\int_0^1 t^8\ dt} = \sqrt{\frac{1}{9}} = \frac{1}{3}$.
   $\|s(t)\| = \sqrt{\int_0^1 s(t)^2\ dt} = \sqrt{\int_0^1 (3t^4 - 1)^2\ dt} = \sqrt{\int_0^1 9t^8 - 6t^4 + 1\ dt} = \sqrt{1 - \frac{6}{5} + 1} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$.

   (b) Find the distance between $r(t)$ and $s(t)$.

   $\|r(t) - s(t)\| = \|-2t^4 + 1\| = \sqrt{\int_0^1 (-2t^4 + 1)^2\ dt} = \sqrt{\int_0^1 4t^8 - 4t^4 + 1\ dt} = \sqrt{\frac{4}{9} - \frac{4}{5} + 1} = \sqrt{\frac{29}{45}}$

   (c) Find the cosine of the angle between $r(t)$ and $s(t)$.

   $\cos(\theta) = \frac{(r(t), s(t))}{\|r(t)\|\|s(t)\|}$.
   $(r(t), s(t)) = \int_0^1 r(t)s(t)\ dt = \int_0^1 3t^8 - t^4\ dt = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$, so $\cos(\theta) = \frac{2/15}{(1/3)(2/\sqrt{5})}$.

4. In the inner product space V from the previous problem, which of the following sets are orthogonal, orthonormal, or neither?

   (a) $\{t^2, t, 1\}$

   This is neither. None of these inner products are 0. For example, $(t^2, t) = \int_0^1 t^3\ dt = \frac{1}{4}$.

   (b) $\{1, 2t - 1\}$

   This is orthogonal. The vectors are orthogonal as $(1, 2t-1) = \int_0^1 2t-1\ dt = 0$. The first one is length 1 since $(1, 1) = \int_0^1 1\ dt = 1$, but the second one is not as $(2t - 1, 2t - 1) = \int_0^1 (2t - 1)^2\ dt = \int_0^1 4t^2 - 4t + 1\ dt = \frac{4}{3} - 2 + 1 \neq 1$.

5. Determine if the following are inner products on $\mathbb{R}^2$.

   (a) $\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) = ac + ad + bc + 2bd$

   This is an inner product. Check the 4 properties.
   Property 1: $(\mathbf{u}, \mathbf{u}) \geq 0$ and equals 0 if and only if $\mathbf{u} = \mathbf{0}$. In this case, $\mathbf{u}$ looks like $\begin{bmatrix} a \\ b \end{bmatrix}$. Then $\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right) = a^2 + ab + ba + 2b^2 = (a + b)^2 + b^2 \geq 0$.

This is equal to 0 if and only if $a+b=0$ and $b=0$ which is true if and only if $a=b=0$ so $\mathbf{u}=\begin{bmatrix}0\\0\end{bmatrix}$.

Property 2: $(\mathbf{u},\mathbf{v})=(\mathbf{v},\mathbf{u})$. If $\mathbf{u}=\begin{bmatrix}a\\b\end{bmatrix}, \mathbf{v}=\begin{bmatrix}c\\d\end{bmatrix}$ then $(\mathbf{u},\mathbf{v})=\left(\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}\right)= ac+ad+bc+2bd=ca+cb+cd+2bd=\left(\begin{bmatrix}c\\d\end{bmatrix},\begin{bmatrix}a\\b\end{bmatrix}\right)=(\mathbf{v},\mathbf{u})$.

Property 3: $(\mathbf{u}+\mathbf{v},\mathbf{w})=(\mathbf{u},\mathbf{w})+(\mathbf{v},\mathbf{w})$. If $\mathbf{u}=\begin{bmatrix}a\\b\end{bmatrix}, \mathbf{v}=\begin{bmatrix}c\\d\end{bmatrix}, \mathbf{w}=\begin{bmatrix}e\\f\end{bmatrix}$ then $(\mathbf{u}+\mathbf{v},\mathbf{w})=\left(\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}\right)=\left(\begin{bmatrix}a+c\\b+d\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}\right)=(a+c)e+(a+c)f+(b+d)e+2(b+d)f=ae+ce+af+cf+be+de+2bf+2df=(ae+af+be+2bf)+(ce+cf+de+2df)=\left(\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}\right)+\left(\begin{bmatrix}c\\d\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}\right)=(\mathbf{u},\mathbf{w})+(\mathbf{v},\mathbf{w})$.

Property 4: $(r\mathbf{u},\mathbf{v})=r(\mathbf{u},\mathbf{v})$. If $\mathbf{u}=\begin{bmatrix}a\\b\end{bmatrix}, \mathbf{v}=\begin{bmatrix}c\\d\end{bmatrix}$ then $(r\mathbf{u},\mathbf{v})=\left(r\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}\right)=\left(\begin{bmatrix}ra\\rb\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}\right)=rac+rad+rbc+2rbd=r(ac+ad+bc+2bd)=r\left(\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}\right)=r(\mathbf{u},\mathbf{v})$.

(b) $\left(\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}\right)=(a+c)^2+(b+d)^2$

This is not an inner product. It does not satisfy properties 3 and 4. To answer this question, you only need to demonstrate that it fails one of the properties, but we will review all 4 properties here.

Property 1: $(\mathbf{u},\mathbf{u})\geq 0$ and equals 0 if and only if $\mathbf{u}=\mathbf{0}$. In this case, $\mathbf{u}$ looks like $\begin{bmatrix}a\\b\end{bmatrix}$. Then $\left(\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}a\\b\end{bmatrix}\right)=(2a)^2+(2b)^2=4(a^2+b^2)$. This is always ≥ 0 and equals 0 if and only if $a=b=0$.

Property 2: $(\mathbf{u},\mathbf{v})=(\mathbf{v},\mathbf{u})$. If $\mathbf{u}=\begin{bmatrix}a\\b\end{bmatrix}, \mathbf{v}=\begin{bmatrix}c\\d\end{bmatrix}$ then $(\mathbf{u},\mathbf{v})=\left(\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}\right)= (a+c)^2+(b+d)^2=(c+a)^2+(d+b)^2=\left(\begin{bmatrix}c\\d\end{bmatrix},\begin{bmatrix}a\\b\end{bmatrix}\right)=(\mathbf{v},\mathbf{u})$.

Property 3: $(\mathbf{u}+\mathbf{v},\mathbf{w})=(\mathbf{u},\mathbf{w})+(\mathbf{v},\mathbf{w})$. If $\mathbf{u}=\begin{bmatrix}a\\b\end{bmatrix}, \mathbf{v}=\begin{bmatrix}c\\d\end{bmatrix}, \mathbf{w}=\begin{bmatrix}e\\f\end{bmatrix}$ then $(\mathbf{u}+\mathbf{v},\mathbf{w})=\left(\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}\right)=\left(\begin{bmatrix}a+c\\b+d\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}\right)=(a+c+e)^2+(b+d+f)^2$. However, $(\mathbf{u},\mathbf{w})+(\mathbf{v},\mathbf{w})=\left(\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}\right)+\left(\begin{bmatrix}c\\d\end{bmatrix},\begin{bmatrix}e\\f\end{bmatrix}\right)=(a+e)^2+(b+f)^2+(c+e)^2+(d+f)^2$. These do not look equal, but we check with a concrete example to make sure. Take $a = b = c = d = e = f = 1$. Then $(a+c+e)^2+(b+d+f)^2 = 18$ and $(a+e)^2+(b+f)^2+(c+e)^2+(d+f)^2 = 16$. This property does not hold.

Property 4: $(r\mathbf{u}, \mathbf{v}) = r(\mathbf{u}, \mathbf{v})$. If $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}, \mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ then $(r\mathbf{u}, \mathbf{v}) = \left(r \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) = \left(\begin{bmatrix} ra \\ rb \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) = (ra + c)^2 + (rb + d)^2$. The other side is $r(\mathbf{u}, \mathbf{v}) = r \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) = r((a + c)^2 + (b + d)^2)$. These are not equal, for example if $r = 0, a = b = c = d = 1$ then $(ra + c)^2 + (rb + d)^2 = 2$ and $r((a + c)^2 + (b + d)^2) = 0$.

6. Let V be an inner product space and $\mathbf{v}$ be a fixed vector in V. Let W be the set of all vectors $\mathbf{w}$ in V such that $(\mathbf{v}, \mathbf{w}) = 0$ (i.e. the set of all vectors which are orthogonal to $\mathbf{v}$). Prove that W is a subspace of V.

   This set is nonempty as $\mathbf{0}$ is in W (see next problem). First we show this is closed under addition. If $\mathbf{w_1}$ and $\mathbf{w_2}$ are in W then $(\mathbf{w_1}, \mathbf{v}) = 0$ and $(\mathbf{w_2}, \mathbf{v}) = 0$. Their sum, $\mathbf{w_1}+\mathbf{w_2}$ is also in W as $(\mathbf{w_1}+\mathbf{w_2}, \mathbf{v}) = (\mathbf{w_1}, \mathbf{v})+(\mathbf{w_2}, \mathbf{v}) = 0+0 = 0$ (note that this uses property 3 of inner products). Next show W is closed under scalar multiplication. If $\mathbf{w}$ is in W then $(\mathbf{w}, \mathbf{v}) = 0$. For any real number r, $r\mathbf{w}$ is also in W because $(r\mathbf{w}, \mathbf{v}) = r(\mathbf{w}, \mathbf{v}) = r(0) = 0$ (note that this uses property 4 of inner products).

7. Let V be an inner product space. Show the following:

   (a) $\|\mathbf{0}\| = 0$

   By property 1 of inner products, $(\mathbf{0}, \mathbf{0}) = 0$ so $\|\mathbf{0}\| = \sqrt{(\mathbf{0}, \mathbf{0})} = 0$.

   (b) $(\mathbf{v}, \mathbf{0}) = 0$ for any $\mathbf{v}$ in V.

   $(\mathbf{v}, \mathbf{0}) = (\mathbf{v}, 2\mathbf{0}) = 2(\mathbf{v}, \mathbf{0})$. Subtracting $(\mathbf{v}, \mathbf{0})$ from both sides gives $0 = (\mathbf{v}, \mathbf{0})$.

   (c) If $(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{v}$ in V then $\mathbf{u} = \mathbf{0}$.

   If $(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{v}$ in V, then $\mathbf{u}$ is orthogonal to all vectors in V. In particular, $\mathbf{u}$ is orthogonal to itself, so $(\mathbf{u}, \mathbf{u}) = 0$. By property 1 of inner products, this only happens if $\mathbf{u} = \mathbf{0}$.

8. Let V be a 3-dimensional inner product space and let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be an orthonormal set of vectors in V.

(a) Show that S is a basis for V.

S is orthonormal so it is orthogonal and all vectors in S have length 1. The vectors in S are nonzero because they have length 1. We proved in class that any set of nonzero orthogonal vectors is linearly independent so S is linearly independent. Then as S is linearly independent and has size 3 and the dimension of V is 3, S must be a basis for V.

(b) If $\mathbf{v}$ is a vector in V with $[\mathbf{v}]_S = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$, find the inner products $(\mathbf{v}, \mathbf{v_1})$, $(\mathbf{v}, \mathbf{v_2})$, and $(\mathbf{v}, \mathbf{v_3})$.

As $[\mathbf{v}]_S = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$, $\mathbf{v} = 2\mathbf{v_1} - 4\mathbf{v_2} + 7\mathbf{v_3}$. Then $(\mathbf{v}, \mathbf{v_1}) = (2\mathbf{v_1} - 4\mathbf{v_2} + 7\mathbf{v_3}, \mathbf{v_1})$. By property 3, this equals $(2\mathbf{v_1}, \mathbf{v_1}) + (-4\mathbf{v_2}, \mathbf{v_1}) + (7\mathbf{v_3}, \mathbf{v_1})$. By property 4, this equals $2(\mathbf{v_1}, \mathbf{v_1}) + (-4)(\mathbf{v_2}, \mathbf{v_1}) + 7(\mathbf{v_3}, \mathbf{v_1})$. S is an orthonormal set, so $(\mathbf{v_1}, \mathbf{v_1}) = \|\mathbf{v_1}\|^2 = 1$ and $(\mathbf{v_2}, \mathbf{v_1}) = 0$ and $(\mathbf{v_3}, \mathbf{v_1}) = 0$. Plugging these in, we get $(\mathbf{v}, \mathbf{v_1}) = 2$. By a similar process, $(\mathbf{v}, \mathbf{v_2}) = -4$ and $(\mathbf{v}, \mathbf{v_3}) = 7$.