Homework 10 Due: Wednesday, October 29

1. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. Use the dot product on $\mathbb{R}^3$ to compute the following.

   (a) The lengths of $\mathbf{u}$ and $\mathbf{v}$, $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

   (b) The distance between $\mathbf{u}$ and $\mathbf{v}$.

   (c) The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

2. Determine if the following sets of vectors in $\mathbb{R}^4$ with the dot product are orthogonal, orthonormal, or neither.

   (a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$

   (b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}$

   (c) $\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}$

3. Let $V = P$ with inner product $(p(t), q(t)) = \int_0^1 p(t)q(t)\ dt$.
   Let $r(t) = t^4, s(t) = 3t^4 - 1$.

   (a) Find $\|r(t)\|$ and $\|s(t)\|$.

   (b) Find the distance between $r(t)$ and $s(t)$.

   (c) Find the cosine of the angle between $r(t)$ and $s(t)$.

4. In the inner product space V from the previous problem, which of the following sets are orthogonal, orthonormal, or neither?

   (a) $\{t^2, t, 1\}$

   (b) $\{1, 2t - 1\}$

5. Determine if the following are inner products on $\mathbb{R}^2$.

   (a) $\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) = ac + ad + bc + 2bd$

(b) $\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) = (a+c)^2 + (b+d)^2$

6. Let V be an inner product space and $\mathbf{v}$ be a fixed vector in V. Let W be the set of all vectors $\mathbf{w}$ in V such that $(\mathbf{v}, \mathbf{w}) = 0$ (i.e. the set of all vectors which are orthogonal to $\mathbf{v}$). Prove that W is a subspace of V.

7. Let V be an inner product space. Show the following:

   (a) $\|\mathbf{0}\| = 0$

   (b) $(\mathbf{v}, \mathbf{0}) = 0$ for any $\mathbf{v}$ in V.

   (c) If $(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{v}$ in V then $\mathbf{u} = \mathbf{0}$.

8. Let V be a 3-dimensional inner product space and let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be an orthonormal set of vectors in V.

   (a) Show that S is a basis for V.

   (b) If $\mathbf{v}$ is a vector in V with $[\mathbf{v}]_S = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix}$, find the inner products $(\mathbf{v}, \mathbf{v_1})$, $(\mathbf{v}, \mathbf{v_2})$, and $(\mathbf{v}, \mathbf{v_3})$.