Homework 1 Due: Wednesday, August 27

1. Find all values of c for which the linear system:

$$x + y + z = 1$$

$$x - y = 3$$

$$x + y + z = c^2$$

$$2x + cz = 4$$

   (a) has no solutions.

   (b) has exactly one solution.

   (c) has infinitely many solutions.

   (d) is consistent.

   This linear system has augmented matrix $\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 3 \\ 1 & 1 & 1 & c^2 \\ 2 & 0 & c & 4 \end{array}\right]$. The following row operations will get the matrix almost all the way to REF: $r_2 - r_1 \to r_2, r_3 - r_1 \to r_3, r_4 - 2r_1 \to r_4, r_4 - r_2 \to r_4, (-1/2)r_2 \to r_2, r_3 \leftrightarrow r_4$. The resulting matrix is $\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1/2 & -1 \\ 0 & 0 & c-1 & 0 \\ 0 & 0 & 0 & c^2-1 \end{array}\right]$. This is not in REF because the last two rows may not have first nonzero entry equal to 1, but it is close enough to REF that we can tell what the solutions will look like. If $c^2 - 1 \neq 0$, then there will be no solutions (look at the last row). So if $c \neq 1, -1$ then there are no solutions. If $c = 1$, the last two rows are zeros and the third column doesn't have a leading 1 so there are infinitely many solutions. If $c = -1$, one more row operation will put the matrix in REF and there will be one solutions ($z = 0, y = -1, x = 2$). The system is consistent if there is a least one solution, so it's consistent for $c \neq -1, -1$. So the answers to $(a) - (d)$ are: (a) $c \neq 1, -1$, (b) $c = -1$, (c) $c = 1$, (d) $c \neq 1, -1$.

2. Find the augmented matrix of the linear system.

$$x + 3y = z + 5$$

$$z - y + x = 7$$

$$4y + 5x = 0$$

$$x + 7z + 1 = 3$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 1 & -1 & 1 & 7 \\ 5 & 4 & 0 & 0 \\ 1 & 0 & 7 & 2 \end{array}\right]$$

3. Find the homogeneous linear system with coefficient matrix $A = \begin{bmatrix} 1 & 0 & 2 & 6 & 0 \\ 0 & 1 & 3 & 0 & 9 \\ 4 & 0 & 0 & 1 & -1 \end{bmatrix}$.

   The system is homogeneous, so all equations are set equal to 0. A is 3×5 so there are 3 equations and 5 unknowns. The equations are:

$$x_1 + 2x_3 + 6x_4 = 0$$

$$x_2 + 3x_3 + 9x_5 = 0$$

$$4x_1 + x_4 - x_5 = 0$$

4. Determine if each matrix is in row echelon form (REF), reduced row echelon form (RREF), or neither.

   (a) $\begin{bmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ RREF

   (b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix}$ Neither

   (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ Neither

   (d) $\begin{bmatrix} 1 & 7 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ REF

5. For what point or points (a, b, c) is the following matrix in row echelon form? reduced row echelon form?

$$\begin{bmatrix} 1 & a & 0 \\ 0 & b & 1 \\ 0 & 0 & c \end{bmatrix}$$

   REF: The points (a, b, c) for which the matrix is in REF are anything of the form $(a, 1, 1), (a, 0, 0)$, or $(a, 1, 0)$. Note that there is no restriction on a.

   RREF: The points (a, b, c) for which the matrix is in RREF are $(0, 1, 0)$ and anything of form $(a, 0, 0)$.

6. What are the possibilities for the number of solutions to a linear system? Explain how you can determine the number of solutions from the REF or RREF of the augmented matrix of the system.

   Three possibilities: 0, 1, or infinite. If the REF/RREF has a row that looks like $\left[\begin{array}{cccc|c} 0 & 0 & \ldots & 0 & 1 \end{array}\right]$ then there are no solutions. If it does not have a row like that, then you can tell by looking at how many columns to the left of the dashed line have leading ones - if all have leading ones there is one solutions, if not then there are an infinite number of solutions.

7. The following augmented matrices are in REF or RREF. Determine the number of solutions to the corresponding linear system. Find all solutions - your answer should be written as a vector.

(a) $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 7 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

One solution - use back substitution to find it. It is $\begin{bmatrix} -3 \\ -12 \\ -7 \\ 3 \end{bmatrix}$.

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

No solutions - last row is equation $0 = 1$.

(c) $\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Infinitely many solutions. Using variables x, y, z, w, the equations are $x + 2y - w = 1$ and $z + 3w = 0$. Columns 2 and 4 do not have leading ones so y, w can be anything and solving for x, z in terms of y, w we get $x = 1-2y+w$ and $z = -3w$. The solutions are all vectors of the form $\begin{bmatrix} 1-2y+w \\ y \\ -3w \\ w \end{bmatrix}$, where y, w can be anything.

8. Find the augmented matrix of each linear system and put it into RREF. Make sure to write down the row operations that you are doing. Use the RREF to find all solutions to the linear systems.

(a)

$$\begin{aligned} x + 2y - 3z &= -5 \\ 3x + 6y - 8z &= -13 \\ -x - 2y + 2z &= 3 \end{aligned}$$

$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -5 \\ 3 & 6 & -8 & -13 \\ -1 & -2 & 2 & 3 \end{array}\right]$

The following row operations will take this to RREF: $r_2 - 3r_1 \to r_2, r_3 + r_1 \to r_3, r_3 + r_2 \to r_3, r_1 + 3r_2 \to r_1$. The RREF is $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Column 2 does not contain a leading 1 so y can be anything, $z = 2$ from row 2, and $x = 1-2y$ so the solutions are all vectors of the form $\begin{bmatrix} 1-2y \\ y \\ 2 \end{bmatrix}$.

(b)

$$3x + 4y - 2z = 2$$
$$2y = 4$$
$$x + y + z = 0$$

$$\left[\begin{array}{ccc:c} 3 & 4 & -2 & 1 \\ 0 & 2 & 0 & 4 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

The following row operations will take this to RREF: $r_1 \leftrightarrow r_3, r_3 - 3r_1 \rightarrow r_3, (1/2)r_2 \rightarrow r_2, r_3 - r_2 \rightarrow r_3, (-1/5)r_3 \rightarrow r_3, r_1 - r_3 \rightarrow r_1, r_1 - r_2 \rightarrow r_1$. The RREF is $\left[\begin{array}{ccc:c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$. These are the equations $x = -2, y = 2, z = 0$ so the solution is $\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$.

9. Suppose A is a 4×4 matrix for which the homogeneous linear system with coefficient matrix A has exactly one solution. Find the RREF of A.

   The augmented matrix of the homogeneous system looks like $[A : 0]$. Row operations will not change the column of zeros on the right, so the RREF of the augmented matrix will be $[B : 0]$ where B is the RREF of A. The system has only one solution so each column of B contains a leading one. B is in RREF so the rest of the entries must all be zeros and the leading ones must move to the right, so B has to be $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

10. Determine if each statement is true or false.

    (a) If A and B are matrices with the same RREF, then A and B are row equivalent to each other.

    True. Let C be RREF of A and B. There are row operations taking A to C and row operations taking C to B so we can do row operations to get from A to B.

(b) Any two linear systems whose coefficient matrices are row equivalent have the same solutions.

False. For example, linear system $x = 4, y = 5$ has coefficient matrix I_2 and the linear system $x + y = 4, x - y = 5$ has coefficient matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ which is row equivalent to I_2. The coffeficient matrices are row equivalent, but the solutions are different.

This statement would be true if it said augemented matrix instead of coefficient matrix. It would also be true if the linear system was homogeneous.

(c) Given a homogeneous linear system of m equations and n unknonws, if $m < n$ then the system has infinitely many solutions.

True. The system has at least one solution because it is homogeneous. Since $m < n$, there are more columns than rows to the left of the dashed line in the augemented linear system so it is not possible to have a leading one in each column and there must be infinitely many solutions.

(d) Given a homogeneous linear system of m equations and n unknonws, if $n < m$ then the system has exactly one solution.

False. For example, the system $x + y = 0, 2x + 2y = 0, 3x + 3y = 0$ has 3 equations and 2 unknowns and infinitely many solutions, since the three equations are really telling us the same thing. Another example would be $x + y + z = 0, x = 0, y + z = 0, x - y - z = 0$ which has 4 equations and 3 unknowns and infinite solutions.