Final Exam Solutions - Section 004

1. Which of the following sets are subspaces of $\mathbb{R}^3$? Circle yes if it is a subspace and no if it is not. (15 pts)

   (a) A plane through the origin in $\mathbb{R}^3$. yes

   (b) A line in $\mathbb{R}^3$ which does not go through the origin. no

   (c) The origin. yes

   (d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \right\}$ no

   (e) A sphere of radius 1 in $\mathbb{R}^3$ centered at the origin. no

   (f) A ball of radius 1 in $\mathbb{R}^3$ centered at the origin (this is the sphere and its interior) no

   (g) The solutions to the linear system $A\mathbf{x} = \mathbf{b}$ where A is a fixed 3×3 matrix and $\mathbf{b}$ is a fixed nonzero vector. no

   (h) The set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $z = xy$. no

   (i) The column space of a 3×5 matrix. yes

   (j) The null space of a 4×3 matrix. yes

2. Let W be the subspace of $\mathbb{R}^4$ which consists of vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $a + c = b + d$.

   (a) Find a basis for W and the dimension of W. (6 pts)

   Solving $a + c = b + d$ for d, we get that W is the set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \\ a+c-b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ spans W. It is also linearly independent, so it is a basis for W and $\dim W = 3$. Note that if we solved for $a, b,$ or c then the basis would look slightly different.

(b) Assuming the dot product on $\mathbb{R}^4$, find a basis for $W^\perp$. (6 pts)

$W^\perp$ is the set of vectors which are orthogonal to all vectors in W. If a vector is orthogonal to the vectors in a basis for W, then it will be orthogonal to all of W so we need to find $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ which are orthogonal to all 3 vectors from the basis for W in part (a). Then

$0 = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = x + w$, $0 = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = y - w$, and

$0 = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = z + w$. We are therefore looking for solutions to the linear system $x + w = 0, y - w = 0, z + w = 0$. The variable w can be anything and $z = -w, y = w, x = -w$ so the vectors in $W^\perp$ are all vectors of the form $\begin{bmatrix} -w \\ w \\ -w \\ w \end{bmatrix}$. This has basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

3. Let $A = \begin{bmatrix} -4 & -7 & 7 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) Find the eigenvalues of A. (6 pts)

$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda + 4 & 7 & -7 \\ 1 & \lambda - 2 & -1 \\ 0 & 0 & \lambda - 3 \end{bmatrix}\right) =$
$(\lambda - 3)((\lambda + 4)(\lambda - 2) - 7) = (\lambda - 3)(\lambda^2 + 2\lambda - 15) = (\lambda - 3)^2(\lambda + 5)$.
The eigenvalues are 3 (with multiplicity 2) and -5 (with multiplicity 1).

(b) For each eigenvalue, find a basis for the associated eigenspace. (8 pts)

$\lambda = 3$: The eigenspace is the solutions to $(3I - A)\mathbf{x} = \mathbf{0}$. This has coefficient matrix $\begin{bmatrix} 7 & 7 & -7 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The RREF of this matrix is $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
If the entries in the eigenvector are a, b, c then b, c can be anything and $a = -b + c$. The eigenspace is all vectors of the form

$$\begin{bmatrix} -b+c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$ This has basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\lambda = -5$: The eigenspace is the solutions to $(-5I - A)\mathbf{x} = \mathbf{0}$. This has coefficient matrix $\begin{bmatrix} -1 & 7 & -7 \\ 1 & -7 & -1 \\ 0 & 0 & -8 \end{bmatrix}$. The RREF of this matrix is $\begin{bmatrix} 1 & -7 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. If the entries in the eigenvector are a, b, c then $c = 0$, b can be anything, and $a = 7b$. The eigenspace is all vectors of the form $\begin{bmatrix} 7b \\ b \\ 0 \end{bmatrix} = b\begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix}$. This has basis $\left\{ \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(c) Is A diagonalizable? Why or why not? (4 pts)

Yes. The multiplicity of each eigenvalue matches the dimension of the eigenspace.

4. Let $L : P_1 \to \mathbb{R}^3$ be the linear transformation $L(at+b) = \begin{bmatrix} a+2b \\ a+b \\ a-b \end{bmatrix}$.

(a) Find the dimension of the kernel L and the dimension of the range of L. (6 pts)

The kernel of L is the set of all $at+b$ in P_1 such that $L(at+b) = \mathbf{0}$. We therefore need $a + 2b = 0, a + b = 0, a - b = 0$. The only solution to this is $a = 0, b = 0$ so the kernel is just the zero vector and $\dim \ker L = 0$. Then $\dim \operatorname{range} L = \dim P_1 - \dim \ker L = 2 - 0 = 2$.

(b) Is L one-to-one? Is L onto? (4 pts)

L is one-to-one because the kernel is just the zero vector. L is not onto because the range of L has dimension 2 so it cannot be all of $\mathbb{R}^3$.

(c) Find the representation of L with respect to S and T where $S = \{5t-1, 2t+3\}$ and $T = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. (8 pts)

Start by plugging the vectors of S into L to get $L(5t-1) = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$ and $L(2t+3) = \begin{bmatrix} 8 \\ 5 \\ -1 \end{bmatrix}$. Then take coordinates with respect to T.

$\begin{bmatrix}3\\4\\6\end{bmatrix} = 2\begin{bmatrix}0\\0\\1\end{bmatrix} + 1\begin{bmatrix}0\\1\\1\end{bmatrix} + 3\begin{bmatrix}1\\1\\1\end{bmatrix}$ so the coordinate vector is $\begin{bmatrix}2\\1\\3\end{bmatrix}$.
$\begin{bmatrix}8\\5\\-1\end{bmatrix} = -6\begin{bmatrix}0\\0\\1\end{bmatrix} - 3\begin{bmatrix}0\\1\\1\end{bmatrix} + 8\begin{bmatrix}0\\0\\1\end{bmatrix}$ so the coordinate vector is $\begin{bmatrix}-6\\-3\\8\end{bmatrix}$.
These are the columns of the representation so the representation is
$\begin{bmatrix}2 & -6\\1 & -3\\3 & 8\end{bmatrix}$.

5. Let $V = \mathbb{R}^3$ with the following inner product:

$$\left(\begin{bmatrix}a\\b\\c\end{bmatrix}, \begin{bmatrix}d\\e\\f\end{bmatrix}\right) = (a-b)(d-e) + 4be + cf$$

(a) Find the length of the vector $\begin{bmatrix}0\\1\\0\end{bmatrix}$. (5 pts)

Use the formula $\|\mathbf{v}\| = \sqrt{(\mathbf{v}, \mathbf{v})}$.
$\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right) = (0-1)(0-1) + 4(1)(1) + (0)(0) = 5$ so the length of
$\begin{bmatrix}0\\0\\1\end{bmatrix}$ is $\sqrt{5}$.

(b) Determine if the set $S = \left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}5\\1\\0\end{bmatrix}\right\}$ is orthogonal, orthonormal, or neither. (8 pts)

Check if each pair of vectors in S is orthogonal.
$\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right) = (0-0)(0-1) + 4(0)(1) + (1)(0) = 0$,
$\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}5\\1\\0\end{bmatrix}\right) = (0-0)(5-1) + 4(0)(1) + (1)(0)$, and
$\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}5\\1\\0\end{bmatrix}\right) = (0-1)(5-1) + 4(1)(1) + (0)(0) = -4 + 4 = 0$. All three pairs are orthogonal, so the set is orthogonal. It is not orthonormal

because the vectors are not all length 1 (we showed the second one was length $\sqrt{5}$ in part (a)).

6. Let $A = \begin{bmatrix} -2 & 0 & 0 \\ -4 & 7 & -4 \\ -1 & 6 & -3 \end{bmatrix}$. Let $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$.

   (a) Prove that the vectors in S are eigenvectors of A and find their associated eigenvalues. (6 pts)

   For each vector $\mathbf{v}$ in S, prove that $A\mathbf{v}$ is a multiple of $\mathbf{v}$.
   $\begin{bmatrix} -2 & 0 & 0 \\ -4 & 7 & -4 \\ -1 & 6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ so the first vector is an eigenvector with eigenvalue 3. $\begin{bmatrix} -2 & 0 & 0 \\ -4 & 7 & -4 \\ -1 & 6 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ so the eigenvalue is -2. $\begin{bmatrix} -2 & 0 & 0 \\ -4 & 7 & -4 \\ -1 & 6 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ so the eigenvalue is 1.

   (b) Find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$. (6 pts)

   The eigenvalues of A are 3, -2, and 1 so $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The associated eigenvectors are the vectors of S, these will be the columns of P so $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix}$.

   (c) Find the inverse of the matrix P from part (b). (6 pts)

   Start with $[P : I]$ and do row operations to get $[I : P^{-1}]$. The row operations $-r_1 \to r_1, r_1 \leftrightarrow r_2, r_3 - r_1 \to r_3, r_3 - r_2 \to r_3, r_1 - 2r_3 \to r_1$ will take P to I. Doing these row operations to I will give you
   $P^{-1} = \begin{bmatrix} -2 & 3 & -2 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.

   (d) Find A^{50}. (6 pts)

   Note: Your answer should be a single matrix. The entries of the matrix do not need to be simplified (they can contain terms like r^{50}).

   $D = P^{-1}AP$ so $A = PDP^{-1}$ and $A^{50} = PD^{50}P^{-1}$. As D is diagonal,
   $D^{50} = \begin{bmatrix} 3^{50} & 0 & 0 \\ 0 & (-2)^{50} & 0 \\ 0 & 0 & 1^{50} \end{bmatrix} = \begin{bmatrix} 3^{50} & 0 & 0 \\ 0 & 2^{50} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$A^{50} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3^{50} & 0 & 0 \\ 0 & 2^{50} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -2 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2^{50} & 0 & 0 \\ -2(3^{50}) + 2 & 3(3^{50}) - 2 & -2(3^{50}) + 2 \\ -2(3^{50}) - (2^{50}) + 3 & 3(3^{50}) - 3 & -2(3^{50}) + 3 \end{bmatrix}.$$

Bonus: Find the determinant of the following 16×16 matrix. (5 pts)
You must show work to get credit.

$$\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}$$

Let A_k denote the $k \times k$ matrix with 1's on the diagonal, -1's right above and right below the diagonal, and 0's elsewhere. When $k = 1$, A_k is the 1×1 matrix with entry 1 so $\det(A_1) = 1$. When $k = 2$, $A_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ so $\det(A_2) = 0$. Assume now that $k \geq 3$. If we do cofactor expansion along the first row, we get $\det(A_k) = (1)\det(A_{k-1}) - (-1)\det(B)$ where B is gotten from A_k by deleting the first row and second column. Then doing cofactor expansion along column 1 of B, $\det(B) = (-1)\det(A_{k-2})$. Putting this together, $\det(A_k) = \det(A_{k-1}) - \det(A_{k-2})$. Since we know $\det(A_1)$ and $\det(A_2)$ we can use this formula to recursively find $\det(A_{16})$. The determinants of A_k from $k = 1$ to $k = 16$ are 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1. The above matrix is A_{16} and $\det(A_{16}) = -1$.