Final Exam Solutions - Section 001

1. Which of the following sets are subspaces of $\mathbb{R}^3$? Circle yes if it is a subspace and no if it is not. (15 pts)

   (a) A line in $\mathbb{R}^3$ which does not go through the origin. no

   (b) A plane through the origin in $\mathbb{R}^3$. yes

   (c) The origin. yes

   (d) A sphere of radius 1 in $\mathbb{R}^3$ centered at the origin. no

   (e) A ball of radius 1 in $\mathbb{R}^3$ centered at the origin (this is the sphere and its interior) no

   (f) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \right\}$ no

   (g) The null space of a 4×3 matrix. yes

   (h) The solutions to the linear system $A\mathbf{x} = \mathbf{b}$ where A is a fixed 3×3 matrix and $\mathbf{b}$ is a fixed nonzero vector. no

   (i) The column space of a 3×5 matrix. yes

   (j) The set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $z = xy$. no

2. Let W be the subspace of $\mathbb{R}^4$ which consists of vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $a + b = c + d$.

   (a) Find a basis for W and the dimension of W. (6 pts)

   Solving $a + b = c + d$ for d, we get that W is the set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \\ a+b-c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ spans W. It is also linearly independent, so it is a basis for W and $\dim W = 3$. Note that if we solved for $a, b,$ or c then the basis would look slightly different.

(b) Assuming the dot product on $\mathbb{R}^4$, find a basis for $W^\perp$. (6 pts)

$W^\perp$ is the set of vectors which are orthogonal to all vectors in W. If a vector is orthogonal to the vectors in a basis for W, then it will be orthogonal to all of W so we need to find $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ which are orthogonal to all 3 vectors from the basis for W in part (a). Then $0 = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = x + w$, $0 = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = y + w$, and $0 = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = z - w$. We are therefore looking for solutions to the linear system $x + w = 0, y + w = 0, z - w = 0$. The variable w can be anything and $z = w, y = -w, x = -w$ so the vectors in $W^\perp$ are all vectors of the form $\begin{bmatrix} -w \\ -w \\ w \\ w \end{bmatrix}$. This has basis $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

3. Let $A = \begin{bmatrix} 1 & 3 & -8 \\ 0 & 2 & 0 \\ -1 & 3 & -6 \end{bmatrix}$.

(a) Find the eigenvalues of A. (6 pts)

$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda - 1 & -3 & 8 \\ 0 & \lambda - 2 & 0 \\ 1 & -3 & \lambda + 6 \end{bmatrix} \right) =$
$(\lambda - 2)((\lambda - 1)(\lambda + 6) - 8) = (\lambda - 2)(\lambda^2 + 5\lambda - 14) = (\lambda - 2)^2(\lambda + 7)$.
The eigenvalues are 2 (with multiplicity 2) and -7 (with multiplicity 1).

(b) For each eigenvalue, find a basis for the associated eigenspace. (8 pts)

$\lambda = 2$: The eigenspace is the solutions to $(2I - A)\mathbf{x} = \mathbf{0}$. This has coefficient matrix $\begin{bmatrix} 1 & -3 & 8 \\ 0 & 0 & 0 \\ 1 & -3 & 8 \end{bmatrix}$. The RREF of this matrix is $\begin{bmatrix} 1 & -3 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
If the entries in the eigenvector are a, b, c then b, c can be anything and $a = 3b - 8c$. The eigenspace is all vectors of the form

$\begin{bmatrix} 3b-8c \\ b \\ c \end{bmatrix} = b\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -8 \\ 0 \\ 1 \end{bmatrix}$. This has basis $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\lambda = -7$: The eigenspace is the solutions to $(-7I - A)\mathbf{x} = \mathbf{0}$. This has coefficient matrix $\begin{bmatrix} -8 & -3 & 8 \\ 0 & -9 & 0 \\ 1 & -3 & -1 \end{bmatrix}$. The RREF of this matrix is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. If the entries in the eigenvector are a, b, c then c is anything, $b = 0$, and $a = c$. The eigenspace is all vectors of the form $\begin{bmatrix} c \\ 0 \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. This has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(c) Is A diagonalizable? Why or why not? (4 pts)

Yes. The multiplicity of each eigenvalue matches the dimension of the eigenspace.

4. Let $L : P_1 \to \mathbb{R}^3$ be the linear transformation $L(at+b) = \begin{bmatrix} a+b \\ a-b \\ a+2b \end{bmatrix}$.

(a) Find the dimension of the kernel L and the dimension of the range of L. (6 pts)

The kernel of L is the set of all $at+b$ in P_1 such that $L(at+b) = \mathbf{0}$. We therefore need $a+b=0, a-b=0, a+2b=0$. The only solution to this is $a=0, b=0$ so the kernel is just the zero vector and $\dim\ker L = 0$. Then $\dim \operatorname{range} L = \dim P_1 - \dim\ker L = 2 - 0 = 2$.

(b) Is L one-to-one? Is L onto? (4 pts)

L is one-to-one because the kernel is just the zero vector. L is not onto because the range of L has dimension 2 so it cannot be all of $\mathbb{R}^3$.

(c) Find the representation of L with respect to S and T where $S = \{2t+3, t-1\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. (8 pts)

Start by plugging the vectors of S into L to get $L(2t+3) = \begin{bmatrix} 5 \\ -1 \\ 8 \end{bmatrix}$ and $L(t-1) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. Then take coordinates with respect to T.

$$\begin{bmatrix}5\\-1\\8\end{bmatrix} = 8\begin{bmatrix}1\\1\\1\end{bmatrix} + -9\begin{bmatrix}1\\1\\0\end{bmatrix} + 6\begin{bmatrix}1\\0\\0\end{bmatrix} \text{ so the coordinate vector is } \begin{bmatrix}8\\-9\\6\end{bmatrix}.$$

$$\begin{bmatrix}0\\2\\-1\end{bmatrix} = -1\begin{bmatrix}1\\1\\1\end{bmatrix} + 3\begin{bmatrix}1\\1\\0\end{bmatrix} + -2\begin{bmatrix}1\\0\\0\end{bmatrix} \text{ so the coordinate vector is } \begin{bmatrix}-1\\3\\-2\end{bmatrix}.$$

These are the columns of the representation so the representation is $\begin{bmatrix}8 & -1\\-9 & 3\\6 & -2\end{bmatrix}$.

5. Let $V = \mathbb{R}^3$ with the following inner product:

$$\left(\begin{bmatrix}a\\b\\c\end{bmatrix}, \begin{bmatrix}d\\e\\f\end{bmatrix}\right) = ad + (b-c)(e-f) + 2cf$$

(a) Find the length of the vector $\begin{bmatrix}0\\0\\1\end{bmatrix}$. (5 pts)

Use the formula $\|\mathbf{v}\| = \sqrt{(\mathbf{v}, \mathbf{v})}$.

$\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = (0)(0) + (0-1)(0-1) + 2(1)(1) = 3$ so the length of $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is $\sqrt{3}$.

(b) Determine if the set $S = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\3\\1\end{bmatrix}\right\}$ is orthogonal, orthonormal, or neither. (8 pts)

Check if each pair of vectors in S is orthogonal.

$\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = (1)(0) + (0-0)(0-1) + 2(0)(1) = 0$,

$\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\3\\1\end{bmatrix}\right) = (1)(0) + (0-0)(3-1) + 2(0)(1) = 0$, and

$\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\3\\1\end{bmatrix}\right) = (0)(0) + (0-1)(3-1) + 2(1)(1) = -2 + 2 = 0$. All three pairs are orthogonal, so the set is orthogonal. It is not orthonormal because the vectors are not all length 1 (we showed the second one was length $\sqrt{3}$ in part (a)).

6. Let $A = \begin{bmatrix} 5 & -3 & -3 \\ 4 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$.

(a) Prove that the vectors in S are eigenvectors of A and find their associated eigenvalues. (6 pts)

For each vector $\mathbf{v}$ in S, prove that $A\mathbf{v}$ is a multiple of $\mathbf{v}$.
$\begin{bmatrix} 5 & -3 & -3 \\ 4 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ so the first vector is an eigenvector with eigenvalue 2. $\begin{bmatrix} 5 & -3 & -3 \\ 4 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} = -3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ so the eigenvalue is -3. $\begin{bmatrix} 5 & -3 & -3 \\ 4 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ so the eigenvalue is 1.

(b) Find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$. (6 pts)

The eigenvalues of A are 2, -3, and 1 so $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The associated eigenvectors are the vectors of S, these will be the columns of P so $P = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 4 \\ 0 & -1 & 0 \end{bmatrix}$.

(c) Find the inverse of the matrix P from part (b). (6 pts)

Start with $[P : I]$ and do row operations to get $[I : P^{-1}]$. The row operations $r_2 - r_1 \to r_2, r_2 + r_3 \to r_2, r_2 \leftrightarrow r_3, -r_2 \to r_2, r_1 - 3r_3 \to r_1$ will take P to I. Doing these row operations to I will give you
$P^{-1} = \begin{bmatrix} 4 & -3 & -3 \\ 0 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

(d) Find A^{50}. (6 pts)
Note: Your answer should be a single matrix. The entries of the matrix do not need to be simplified (they can contain terms like r^{50}).

$D = P^{-1}AP$ so $A = PDP^{-1}$ and $A^{50} = PD^{50}P^{-1}$. As D is diagonal,
$D^{50} = \begin{bmatrix} 2^{50} & 0 & 0 \\ 0 & (-3)^{50} & 0 \\ 0 & 0 & 1^{50} \end{bmatrix} = \begin{bmatrix} 2^{50} & 0 & 0 \\ 0 & 3^{50} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$A^{50} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 4 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2^{50} & 0 & 0 \\ 0 & 3^{50} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -3 & -3 \\ 0 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} 4(2^{50}) - 3 & -3(2^{50}) + 3 & -3(2^{50}) + 3 \\ 4(2^{50}) - 4 & -3(2^{50}) + 4 & -3(2^{50}) - (3^{50}) + 4 \\ 0 & 0 & 3^{50} \end{bmatrix}.$$

Bonus: Find the determinant of the following 18×18 matrix. (5 pts)
You must show work to get credit.

$$\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}$$

Let A_k denote the $k \times k$ matrix with 1's on the diagonal, -1's right above and right below the diagonal, and 0's elsewhere. When $k = 1$, A_k is the 1×1 matrix with entry 1 so $\det(A_1) = 1$. When $k = 2$, $A_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ so $\det(A_2) = 0$. Assume now that $k \geq 3$. If we do cofactor expansion along the first row, we get $\det(A_k) = (1)\det(A_{k-1}) - (-1)\det(B)$ where B is gotten from A_k by deleting the first row and second column. Then doing cofactor expansion along column 1 of B, $\det(B) = (-1)\det(A_{k-2})$. Putting this together, $\det(A_k) = \det(A_{k-1}) - \det(A_{k-2})$. Since we know $\det(A_1)$ and $\det(A_2)$ we can use this formula to recursively find $\det(A_{18})$. The determinants of A_k from $k = 1$ to $k = 18$ are 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1. The above matrix is A_{18} and $\det(A_{18}) = 1$.