Exam 3 Solutions

1. Let V be an inner product space and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V. Suppose that $\|\mathbf{u}\| = \sqrt{3}$, $\|\mathbf{v}\| = 4$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{6}$. Compute the following inner products.
The following may be useful: $\sin(\frac{\pi}{6}) = \frac{1}{2}$ and $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

   (a) $(\mathbf{u}, \mathbf{u})$ and $(\mathbf{v}, \mathbf{v})$ (6 pts)

   $\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}$ so $(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|^2 = 3$. Similarly, $(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|^2 = 16$.

   (b) $(\mathbf{u}, \mathbf{v})$ (6 pts)

   If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\cos(\theta) = \frac{(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|\|\mathbf{v}\|}$. Plugging in values for $\cos(\theta)$ and the lengths, this becomes $\frac{\sqrt{3}}{2} = \frac{(\mathbf{u},\mathbf{v})}{4\sqrt{3}}$. Solving for $(\mathbf{u}, \mathbf{v})$ we get that $(\mathbf{u}, \mathbf{v}) = 6$.

   (c) $(\mathbf{u} + \mathbf{v}, 2\mathbf{u} - \mathbf{v})$ (8 pts)

   Use the properties of inner products to write this in terms of the inner products computed in parts (a) and (b).
   $(\mathbf{u} + \mathbf{v}, 2\mathbf{u} - \mathbf{v}) = 2(\mathbf{u}, \mathbf{u}) - (\mathbf{u}, \mathbf{v}) + 2(\mathbf{v}, \mathbf{u}) - (\mathbf{v}, \mathbf{v}) =$
   $2(\mathbf{u}, \mathbf{u}) + (\mathbf{u}, \mathbf{v}) - (\mathbf{v}, \mathbf{v}) = 2(3) + 6 - 16 = -4$.

2. Let W be a subspace of the inner product space $\mathbb{R}^4$ with the dot product.
Suppose W has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$.

   (a) Find an orthonormal basis for W. (12 pts)

   Use Gram-Schmidt to first find an orthogonal basis. Label the vectors in the basis for W as $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$. The vectors in the orthogonal basis will be $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. The first vector is $\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. The second is

   $\mathbf{v_2} = -\frac{(\mathbf{v_1},\mathbf{u_2})}{(\mathbf{v_1},\mathbf{v_1})}\mathbf{v_1} + \mathbf{u_2} = -\frac{6}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -2 \end{bmatrix}$. The third is

   $\mathbf{v_3} = -\frac{(\mathbf{v_1},\mathbf{u_3})}{(\mathbf{v_1},\mathbf{v_1})}\mathbf{v_1} - \frac{(\mathbf{v_2},\mathbf{u_3})}{(\mathbf{v_2},\mathbf{v_2})}\mathbf{v_2} + \mathbf{u_3} = -\frac{9}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{9}{9}\begin{bmatrix} 2 \\ -1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$.

The orthogonal basis is $\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\2\\1\\-1\end{bmatrix}\right\}$. Divide each vector by its length to get the orthonormal basis $\left\{\begin{bmatrix}1/\sqrt{3}\\0\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}, \begin{bmatrix}2/3\\-1/3\\0\\-2/3\end{bmatrix}, \begin{bmatrix}0\\2/\sqrt{6}\\1/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}\right\}$.

(b) Is the vector $\begin{bmatrix}-2\\0\\1\\1\end{bmatrix}$ in $W^{\perp}$? Why or why not? (6 pts)

To check if this vector is in $W^{\perp}$, we just need to check if it is orthogonal to the three basis vectors (this will guarantee it is orthogonal to all of W). This can be done with either the original basis or the basis found in (a). Using the original basis, we have $\begin{bmatrix}-2\\0\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\\1\end{bmatrix} = 0$, and $\begin{bmatrix}-2\\0\\1\\1\end{bmatrix} \cdot \begin{bmatrix}4\\-1\\2\\0\end{bmatrix} = -6$. We do not need to compute the third dot product, we can see from the second one that the vector is not in $W^{\perp}$.

3. Let $L : M_{nn} \to M_{nn}$ be the function $L(A) = A^T A$. Is L a linear transformation? Why or why not? (10 pts)

This is not a linear transformation. It does not satisfy either of the properties of a linear transformation. You only need prove that one of the two properties is not satisfied, but here we will show both.

The first property of linear transformations is that $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$. In this case, if A, B are two vectors in M_{nn} then $L(A+B) = (A+B)^T(A+B) = (A^T+B^T)(A+B) = A^TA+B^TA+A^TB+B^TB$ and $L(A) + L(B) = A^TA + B^TB$. These are not equal. For example if $A = B = I$ then $L(I + I) = 4I$ and $L(I) + L(I) = 2I$. The second property of linear transformations is $L(r\mathbf{v}) = rL(\mathbf{v})$. If A is in M_{nn} and r is a real number, then $L(rA) = (rA)^T(rA) = r^2A^TA$ and $rL(A) = rA^TA$. Again, these are not equal.

4. Let $L : P_3 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(at^3 + bt^2 + ct + d) = \begin{bmatrix} a - b + c \\ d + 2b - 2c \\ b - c \end{bmatrix}.$$

(a) Find a basis for the kernel of L. (8 pts)

The kernel of L is all polynomials $at^3 + bt^2 + ct + d$ with $a - b + c = 0, d + 2b - 2c = 0$ and $b - c = 0$. From the third equation, we have that $b = c$ and plugging this into the first and second equations we get that $a = 0, d = 0$. Hence the kernel of L is all polynomials of the form $bt^2 + bt$. This has basis $\{t^2 + t\}$.

(b) Find the dimension of the range of L. Is L onto? (8 pts)

The dimension of the range can be computing using the formula $\dim \ker L + \dim \operatorname{range} L = \dim P_3$. Then $\dim P_3 = 4$ and $\dim \ker L = 1$, so $\dim \operatorname{range} L = 3$. The range of L is a 3-dimensional subspace of $\mathbb{R}^3$ so it must be all of $\mathbb{R}^3$ and L is onto.

Another way to do this is to compute the range directly. The range is all vectors in $\mathbb{R}^3$ of the form $\begin{bmatrix} a - b + c \\ d + 2b - 2c \\ b - c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. This gives us a spanning set for the range. The third vector is the negative of the second so we an delete the third vector without changing the span and the resulting set is linearly independent. Hence the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the range so the dimension of the range is 3 and $\operatorname{range} L = \mathbb{R}^3$ so L is onto.

(c) Find the representation of L with respect to S and T where $S = \{1, t, t^2, t^3\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (10 pts)

$L(1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $L(t) = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $L(t^2) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $L(t^3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. As T is the standard basis for $\mathbb{R}^3$, taking coordinates with respect to T does not change the vectors so the representation of L with respect to S and T is $\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & -2 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$.

5. Let $L : V \to V$ be a linear transformation. Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a basis for V. Suppose we know the following:

$$L(\mathbf{v_1}) = \mathbf{v_1} + \mathbf{v_3}$$

$$L(\mathbf{v_2}) = \mathbf{v_1} + 2\mathbf{v_2} + 3\mathbf{v_3}$$

$$L(\mathbf{v_3}) = 2\mathbf{v_3}$$

(a) Find $L(2\mathbf{v_1} - \mathbf{v_2})$. (6 pts)

Using the properties of linear transformations, $L(2\mathbf{v_1} - \mathbf{v_2}) = 2L(\mathbf{v_1}) - L(\mathbf{v_2}) = 2(\mathbf{v_1} + \mathbf{v_3}) - (\mathbf{v_1} + 2\mathbf{v_2} + 3\mathbf{v_3}) = \mathbf{v_1} - 2\mathbf{v_2} - \mathbf{v_3}$.

(b) Find the representation of L with respect to S. (8 pts)

From the three equations given in the problem, $[L(\mathbf{v_1})]_S = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $[L(\mathbf{v_2})]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $[L(\mathbf{v_3})]_S = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. Hence the representation with respect to S is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 2 \end{bmatrix}$.

(c) Prove that L is invertible and find $L^{-1}(\mathbf{v_3})$. (12 pts)

The easiest way to show L is invertible is to show that the representation of L found in part (b) is an invertible matrix. The determinant of this matrix 4 (nonzero), so it is invertible and so is L.

One way find $L^{-1}(\mathbf{v_3})$ is to apply L^{-1} to both sides of the equation $L(\mathbf{v_3}) = 2\mathbf{v_3}$. This gives that $\mathbf{v_3} = L^{-1}(2\mathbf{v_3}) = 2L^{-1}(\mathbf{v_3})$. Dividing by 2 we get that $L^{-1}(\mathbf{v_3}) = \frac{1}{2}\mathbf{v_3}$.

The other way to do this is to find the inverse of the representation found in part (b). The inverse of that matrix is $\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ -1/2 & -1/2 & 1/2 \end{bmatrix}$. This is the representation of L^{-1} with respect to S so $[L^{-1}(\mathbf{v_3})]_S =$
$\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ -1/2 & -1/2 & 1/2 \end{bmatrix} [\mathbf{v_3}]_S = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$ so
$L^{-1}(\mathbf{v_3}) = 0\mathbf{v_1} + 0\mathbf{v_2} + \frac{1}{2}\mathbf{v_3} = \frac{1}{2}\mathbf{v_3}$.