Exam 2 Solutions

1. Let V be the set of all real numbers with operations $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v}$ and $c \odot \mathbf{u} = |c|\mathbf{u}$ (where $|c|$ is the absolute value of c). Prove that V with the operations $\oplus$ and $\odot$ is NOT a vector space by finding a property from the definition of a vector space which is not satisfied. (10 pts)

   Property 6 is the only property which is not satisfied. $(c+d) \odot \mathbf{u} = |c+d|\mathbf{u}$ and $c \odot \mathbf{u} \oplus d \odot \mathbf{u} = |c|\mathbf{u} + |d|\mathbf{u} = (|c| + |d|)\mathbf{u}$. This property fails because $|c+d|$ is not always equal to $|c| + |d|$. For example, if $c = 1, d = -1$ then $|c+d| = 0$ and $|c| + |d| = 2$.

2. Let W be the set of all 2×2 matrices with determinant 0. Is W a subspace of M_{22}? Why or why not? (10 pts)

   W is not a subspace. It is nonempty and closed under scalar multiplication but it is not closed under addition. For example, the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both in W but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not in W.

3. Let W be the subspace of P_3 which consists of all polynomials of the form $p(t) = at^3 + bt^2 + ct + d$ with $a + d = 2b$. Find a basis for W and $\dim W$. (10 pts)

   If we solve $a + d = 2b$ for d and plug this into $at^3 + bt^2 + ct + d$ we get that the polynomials in W have the form $at^3 + bt^2 + ct + (2b - a) = a(t^3 - 1) + b(t^2 + 2) + ct$. The polynomials in W are therefore all linear combinations of $\{t^3 - 1, t^2 + 2, t\}$ so this set is a spanning set for W. It is also linearly independent because if $a(t^3 - 1) + b(t^2 + 2) + ct = 0$ then a, b, c must all be 0. Therefore $\{t^3 - 1, t^2 + 2, t\}$ is a basis for W. The dimension of W is the size of a basis so $\dim W = 3$.

   Note: If you solved for b or a instead of d, you would get the bases $\{t^3 + \frac{1}{2}t^2, t, \frac{1}{2}t^2 + 1\}$ or $\{2t^3 + t^2, t, -t^3 + 1\}$.

4. Let $S = \left\{ \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 & 0 \end{bmatrix} \right\}$.

   (a) Find a basis for span S. What is the dimension of span S? (12 pts)

   There are a few ways to do this. One method is to take a linear combination of the vectors in S and set them equal $\mathbf{0}$ to get a linear system. This is $a \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 2 & 1 & -1 & 1 \end{bmatrix} + c \begin{bmatrix} 3 & 2 & -2 & 1 \end{bmatrix} + d \begin{bmatrix} 4 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ which gives us the linear system

$a+2b+3c+4d=0, b+2c=0, -b-2c=0, a+b+c=0$. The coefficient matrix of this system is $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. The row operations $r_2+r_3 \to r_3, r_4-r_1 \to r_4, r_4+r_2 \to r_4$ give us the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$. This is not quite in REF, but from here we can see that the leading ones in REF will be in columns 1, 2, 4 so the first, second, and fourth vectors of S are a basis for span S. This basis is $\{\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 & 0 \end{bmatrix}\}$. The dimension of span S is 3.

Another method is to put the vectors in S in as the rows of a matrix to get $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 3 & 2 & -2 & 1 \\ 4 & 0 & 0 & 0 \end{bmatrix}$. The span of S is the row space of this matrix and row operations do not change the row space. So we can take the nonzero rows of REF or RREF to as our basis. The RREF is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so the basis is $\{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}\}$.

(b) Circle yes or no. You do not need to explain your answer. (2 pts each)

Does S span $\mathbb{R}_4$? no

Is S linearly independent? no

Does S contain a basis for $\mathbb{R}_4$? no

Is S contained in a basis for $\mathbb{R}_4$? no

5. Let V be a 2-dimensional space with basis $S = \{\mathbf{v_1}, \mathbf{v_2}\}$. Let $T = \{\mathbf{w_1}, \mathbf{w_2}\}$ where $\mathbf{w_1} = \mathbf{v_1} - \mathbf{v_2}$ and $\mathbf{w_2} = 2\mathbf{v_1} + 3\mathbf{v_2}$.

(a) Show that T is also a basis for V. (12 pts)

Start by showing T is linearly independent. If $a\mathbf{w_1} + b\mathbf{w_2} = \mathbf{0}$, then $\mathbf{0} = a(\mathbf{v_1} - \mathbf{v_2}) + b(2\mathbf{v_1} + 3\mathbf{v_2}) = (a+2b)\mathbf{v_1} + (-a+3b)\mathbf{v_2}$ The set $S = \{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent, so $a+2b=0$ and $-a+3b=0$. The only solution to this system of linear equations is $a=0, b=0$ so T is linearly independent. As T is a linearly independent set of size 2 in a 2-dimensional vector space V, it must be a basis for V.

(b) Find the transition matrix $P_{S \leftarrow T}$ from T to S. (8 pts)

The columns of P are $[\mathbf{w_1}]_S$ and $[\mathbf{w_2}]_S$. As $\mathbf{w_1} = \mathbf{v_1} - \mathbf{v_2}$ we get that $[\mathbf{w_1}]_S = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Similarly, $\mathbf{w_2} = 2\mathbf{v_1} + 3\mathbf{v_2}$ so $[\mathbf{w_2}]_S = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Then $P_{S \leftarrow T} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

(c) If $\mathbf{v}$ is a vector in V with $[\mathbf{v}]_T = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, what is $[\mathbf{v}]_S$? (8 pts)

There are two ways to do this problem. One possibility is to use the identity $[\mathbf{v}]_S = P_{S \leftarrow T}[\mathbf{v}]_T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -8 \\ -17 \end{bmatrix}$. The other way is to use that $\mathbf{v} = 2\mathbf{w_1} - 5\mathbf{w_2} = 2(\mathbf{v_1} - \mathbf{v_2}) + (-5)(2\mathbf{v_1} + 3\mathbf{v_2}) = -8\mathbf{v_1} - 17\mathbf{v_2}$. So $[\mathbf{v}]_S = \begin{bmatrix} -8 \\ -17 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 3 & 6 & 2 & -1 & -4 \\ 1 & 2 & 3 & 0 & -2 \\ -2 & -4 & 1 & 1 & 2 \\ 5 & 10 & 1 & -2 & -6 \\ 3 & 6 & 6 & 1 & 2 \\ 2 & 4 & -5 & -3 & -8 \end{bmatrix}$. The RREF of A is $\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find the rank and nullity of A. (4 pts)

The rank is 3 and the nullity is 2. The rank is the number of leading ones in RREF and the nullity is the number of columns in RREF without leading ones.

(b) Find a basis for the column space of A. (5 pts)

$\left\{ \begin{bmatrix} 3 \\ 1 \\ -2 \\ 5 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 6 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -2 \\ 1 \\ -3 \end{bmatrix} \right\}$. There are leading ones in columns 1,3,4 of RREF so columns 1, 3, 4 of A are a basis for the column space of A.

(c) Find a basis for the row space of A. (5 pts)

$\{\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 5 \end{bmatrix}\}$. Row operations do not change the row space so the nonzero rows of RREF are a basis for the row space.

(d) Find a basis for the null space of A. (8 pts)

Using the variables a, b, c, d, e, columns 2 and 5 of RREF do not have leading ones so b and e can be anything. The equations from RREF are

$a + 2b + e = 0$, $c - e = 0$, $d + 5e = 0$ so $a = -2b - e, c = e, d = -5e$. The solutions to $A\mathbf{x} = \mathbf{0}$ are all vectors of the form

$$\begin{bmatrix} -2b - e \\ b \\ e \\ -5e \\ e \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} -1 \\ 0 \\ 1 \\ -5 \\ 1 \end{bmatrix}.$$

The set $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -5 \\ 1 \end{bmatrix} \right\}$ spans the null space of A and is linearly independent so it is a basis for the null space.